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# Lecture 9: Privacy-Enhancing Technologies-3 -Secure Multiparty Computation

COMP 6712 Advanced Security and Privacy

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2023/3/18

# Roadmap

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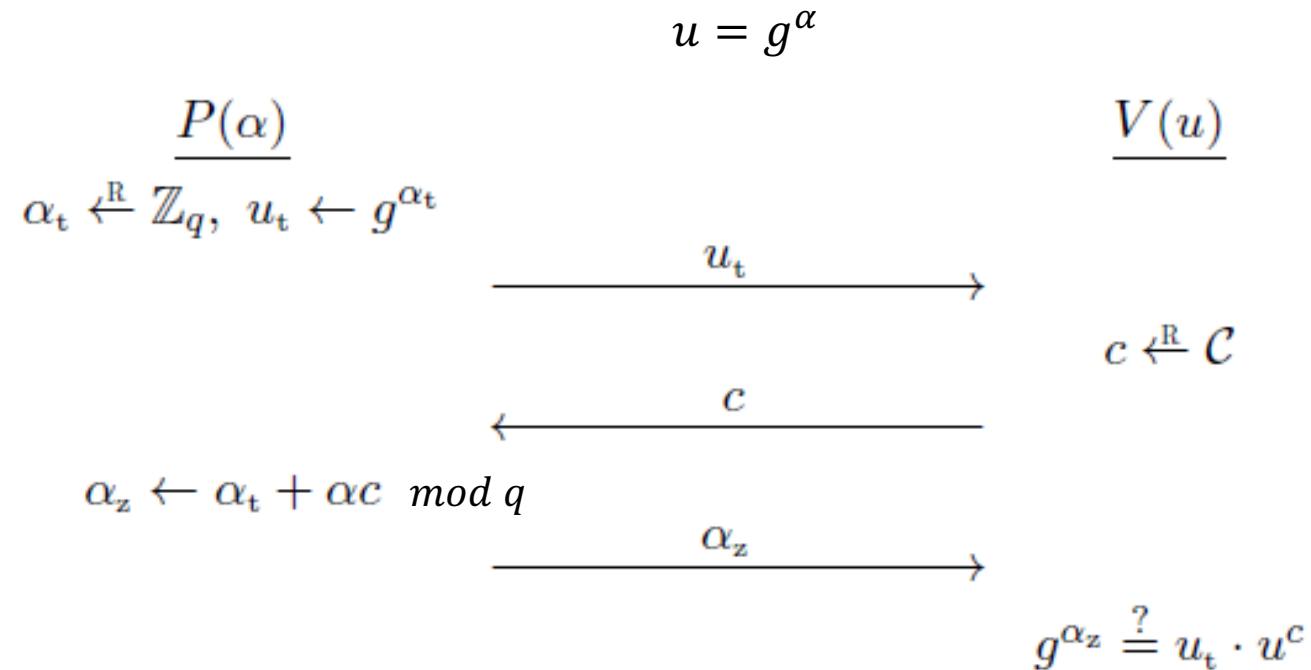
- Recall zero-knowledge proof
- Introduction to Secure Multiparty computation (MPC)
- Yao's Garbled Circuits and GMW protocol
- Practical MPC: Private Set Intersection

# Recall: Zero-knowledge proof

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- Identification protocol and signature
- Sigma protocol
- Zero-knowledge proof
  - Non-interactive ZKP
  - zkSNARK

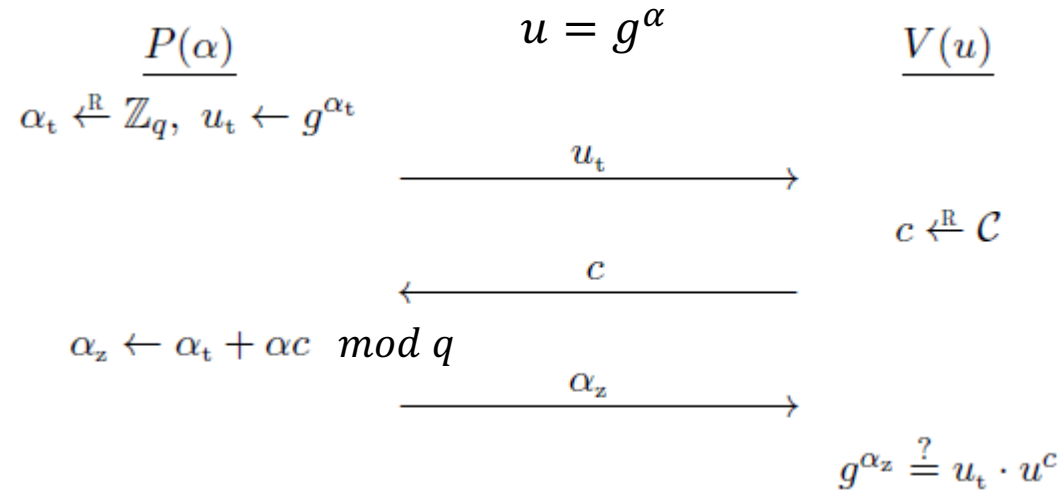
# Schnorr Identification



- Challenge space  $\mathcal{C} = \mathbb{Z}_q$
- Conversation:  $(u_t, c, \alpha_z)$  is said to be valid if the verification passes

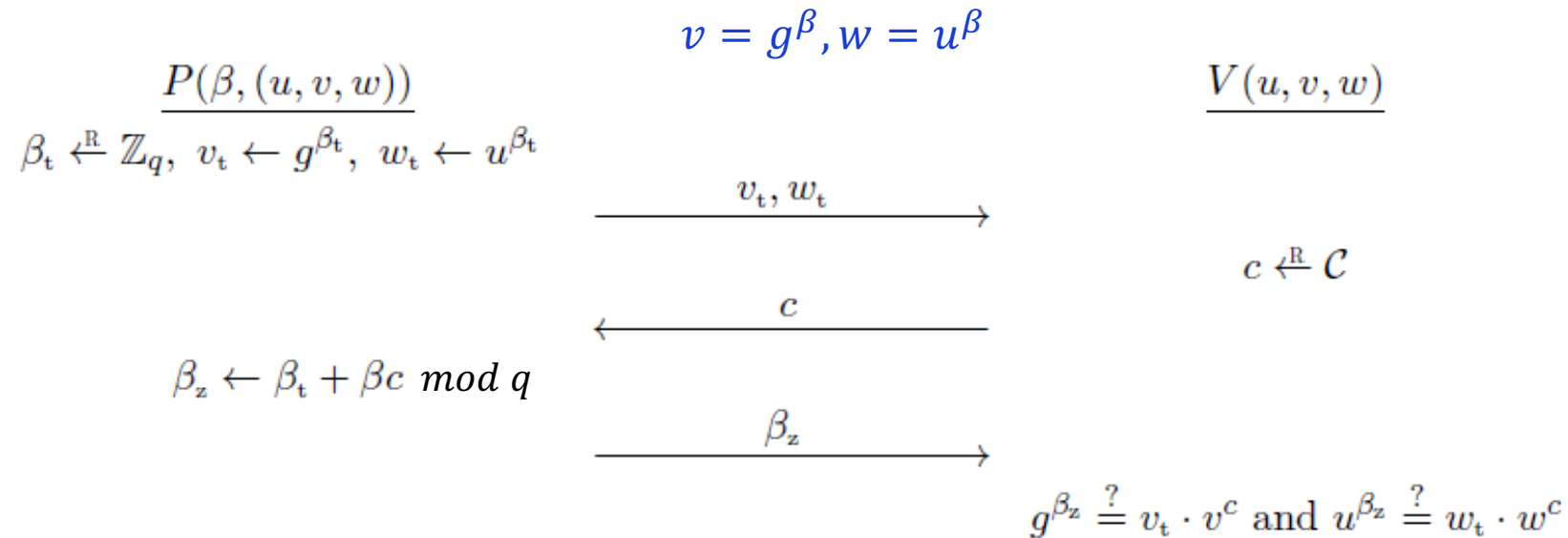
# Schnorr Identification

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- **Correctness(Completeness):** If P and V execute the protocol honestly, the proof is accepted.
- **Soundness (proof-of-knowledge):** If the proof is accepted, we can extract the witness (discrete log)  $\alpha$
- **Honest verifier zero-knowledge** says that: **without knowing** the witness (discrete logarithm), we can generate (simulate) the valid transaction efficiently

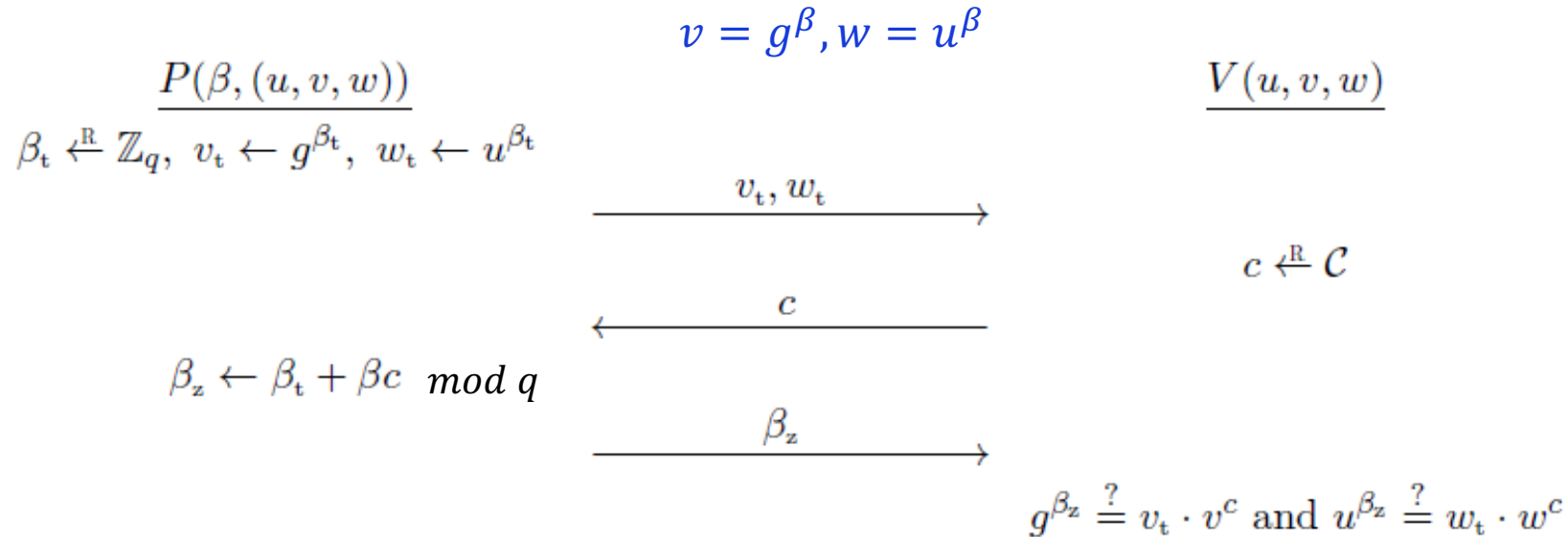
# Identification for Decisional Diffie-Hellman $ID_{DDH}$



Given  $(g, u, v = g^\beta, w = u^\beta)$  with witness  $\beta$ , P wants to prove that it knows  $\beta$

# Identification for Decisional Diffie-Hellman (DDH)

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$$\beta_z \leftarrow \mathbb{Z}_q, c \leftarrow \mathbb{Z}_q, v_t = \frac{g^{\beta_z}}{v^c}, u_t = g^{\beta_z} / u^c$$

# Assignment 2

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- Task 1: zero knowledge proof for that
  - $(c_1, c_2) = (g^\beta, u^\beta \cdot g^1)$  and  $(d_1, d_2) = (g^\gamma, u^\gamma \cdot g^0)$  are the encryption of 1 and 0 respectively
  - Hint: use the proof for DDH tuple
- Task 2: Implement the proof
- Write a report about this
- submit via Blackboard, Deadline: 10 Apr. 11:00 pm



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# Multiparty Computation (MPC)

# Our aim

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- 1 Secure computation:** Concepts & definitions
- 2 General constructions:** Yao's protocol, and GMW
- 3 Custom protocol:** private set intersection

# Secure computation examples: Millionaires Problem

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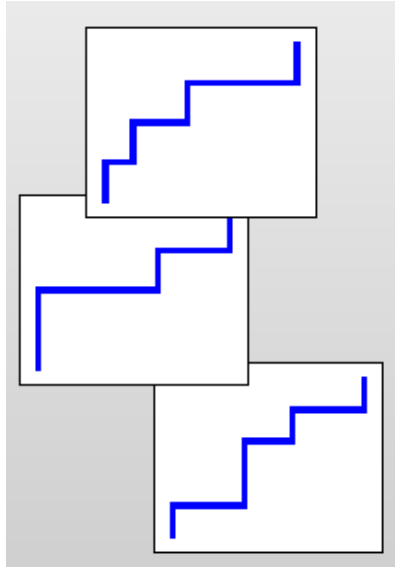


- Alice has money  $x$
- Bob has money  $y$
- $x > y$  or not (but do not want to leak  $x$  or  $y$  to each other )

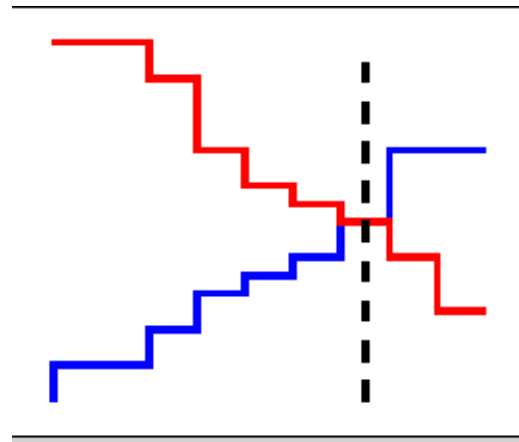
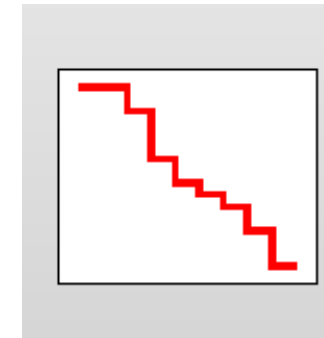
# Secure computation examples: Sugar Bidding

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Farmers

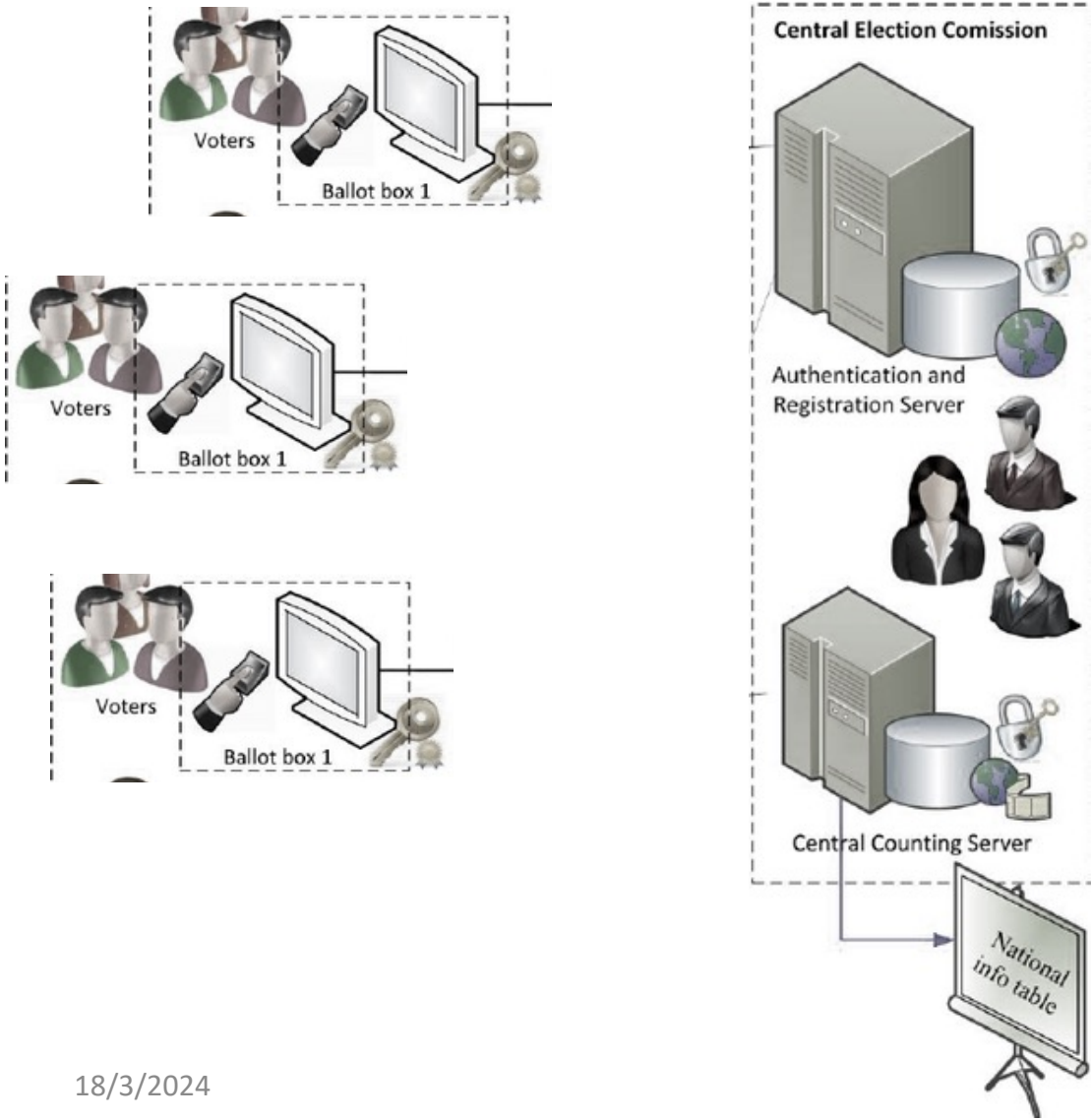


Purchaser



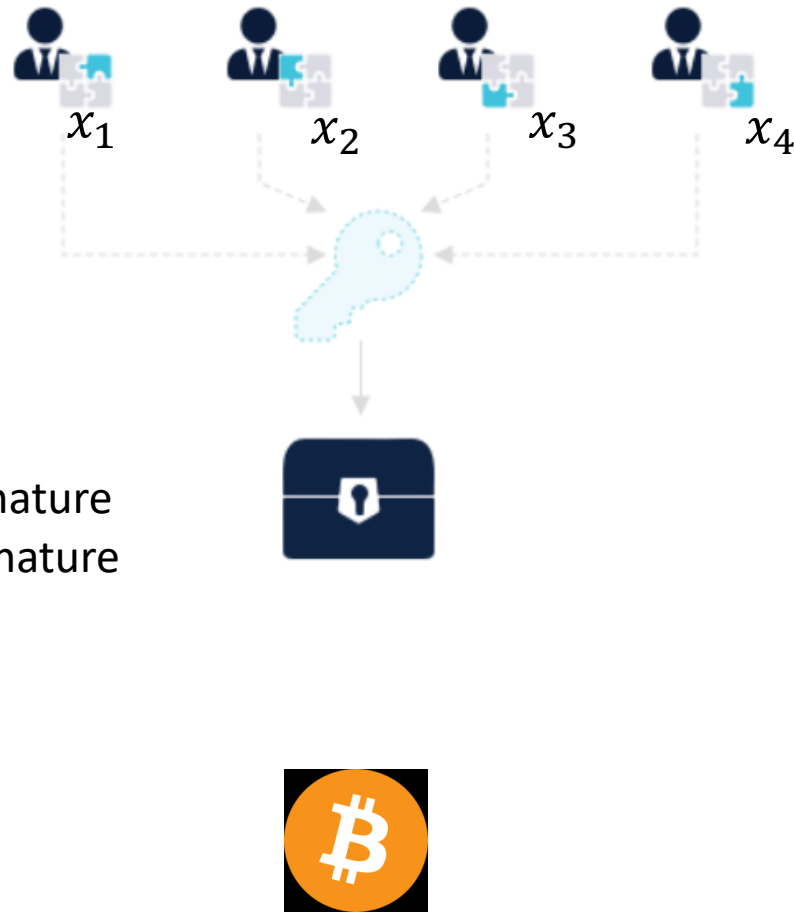
- Farmers make bids (“at price  $X$ , I will produce  $Y$  amount”)
- Purchaser bids (“at price  $X$ , I will buy  $Y$  amount”)
- **Market clearing price (MCP):** price at which total supply = demand

# Secure computation examples: voting



- Secure electronic voting is simply computation of the addition function

# Secure computation examples: Distribute signature



ECDSA Signature  
or RSA signature

- Distribute (ECDSA) signature
- Split the secret signing key into several parts
- such that only they work together can generate the final signature

# Secure computation examples: Ad conversion

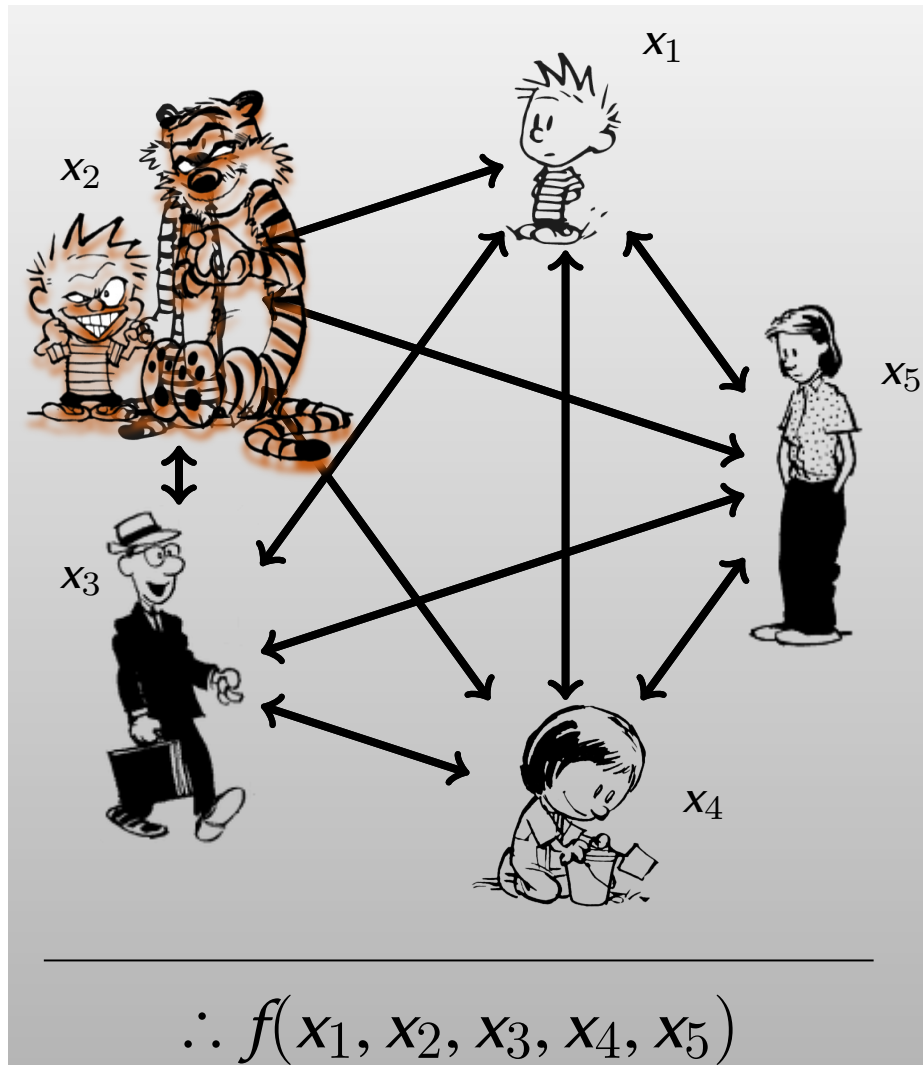
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| Ad impressions         |   | In-store purchases           |
|------------------------|---|------------------------------|
| alice@gmail.com        |    | albert@gmail.com \$80K       |
| <b>bob@gmail.com</b>   |   | <b>bob@gmail.com \$160K</b>  |
| charlie@gmail.com      |   | caroline@gmail.com \$99K     |
| dianne@gmail.com       |   | <b>edwin@gmail.com \$99K</b> |
| <b>edwin@gmail.com</b> |   | felipe@gmail.com \$85K       |
| frank@gmail.com        |   | <b>frank@gmail.com \$77K</b> |
| gina@gmail.com         |   | hilda@gmail.com \$113K       |
|                        |  |                              |

```
SELECT SUM(amount)
FROM ads, purchases
WHERE ads.email = purchases.email
```

- Computed with secure computation by Google and its customers

# Secure computation



## Premise:

- Mutually distrusting parties, each with a private input
  - Learn the result of agreed-upon computation
  - E.g, Millionaires Problem, sugar bidding, Ad conversion...
- 
- Security
    - Privacy (“learn no more than” prescribed output)
    - Input independence
    - Etc...



Two or more parties want to perform some joint computation, while guaranteeing “security” against “adversarial behavior”.

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What does it mean to “security” when  
computing  $f$ ?

Or How do we define security here?

# Security lists for Bidding

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## Consider a secure secret Sugar bidding

- An adversary may wish to learn the bids of all parties – to prevent this, require **PRIVACY**
- An adversary may wish to win with a lower bid– to prevent this, require **CORRECTNESS**
- But, the adversary may also wish to ensure that it always gives the highest bid – to prevent this, require **INDEPENDENCE OF INPUTS**
- An adversary may try to abort the execution if its bid is not the highest – require **FAIRNESS**

# General security requirement

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- **Privacy:** only the output is revealed
- **Correctness:** the function is computed correctly
- **Independence of inputs:** parties cannot choose inputs based on others' inputs
- **Fairness:** if one party receives output, all receive output

# Defining security

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- Option 1: analyze security concerns for each specific problem
  - Bidding: as in previous slide
  - E-voting: privacy, correctness and fairness only?
- Problems:
  - How do we know that all concerns are covered?
  - Definitions are application dependent and need to be redefined from scratch for each task

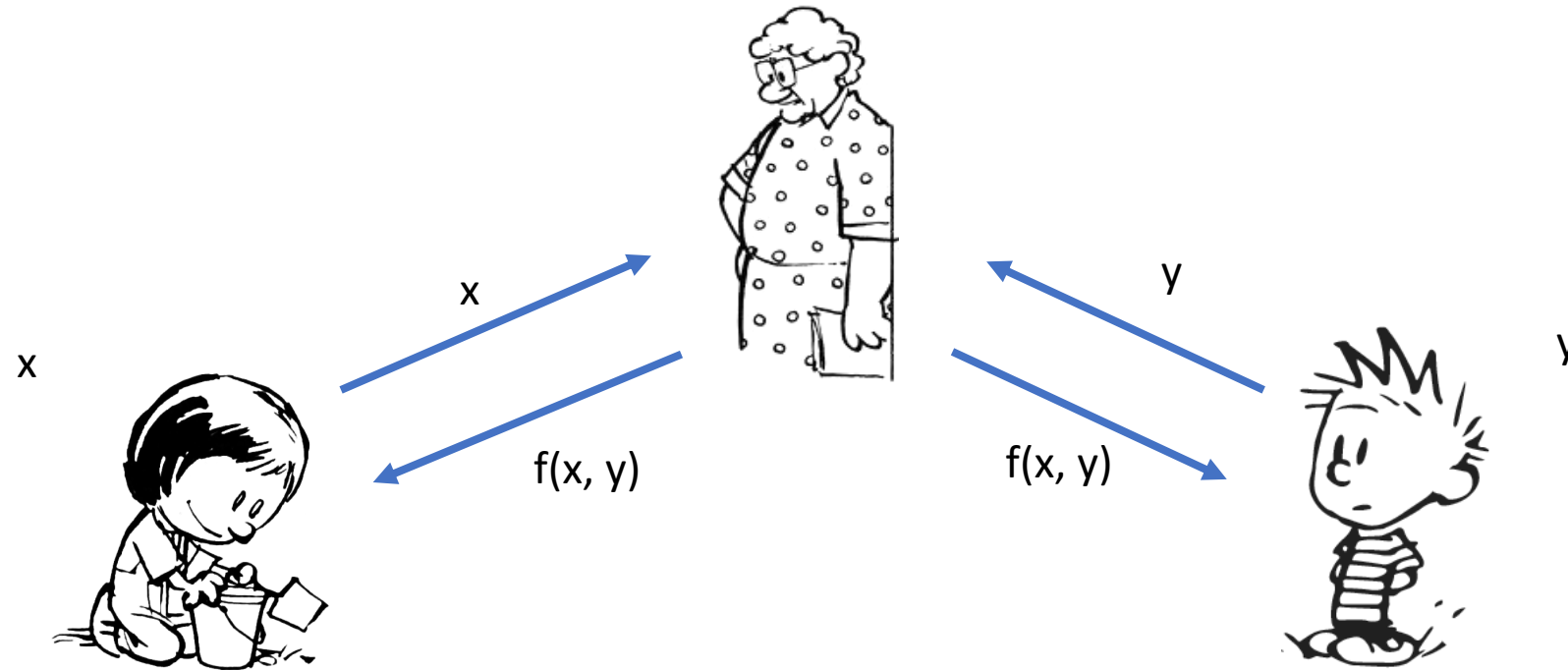
# Defining security

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- Option 2: **general definition** that captures all (most) secure computation tasks
- Properties of any such definition
  - Well-defined adversary model
  - Well-defined execution setting
  - Security guarantees are clear and simple to understand
- How???

# Defining security: ideal world

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- What can a **corrupt party** do in this **ideal world**?
  - Choose any input  $y$  (independent of  $x$ )
  - Learn only  $f(x, y)$ , and nothing more
  - Cause honest party to learn  $f(x, y)$

# Real-ideal paradigm [GoldwasserMicali84]

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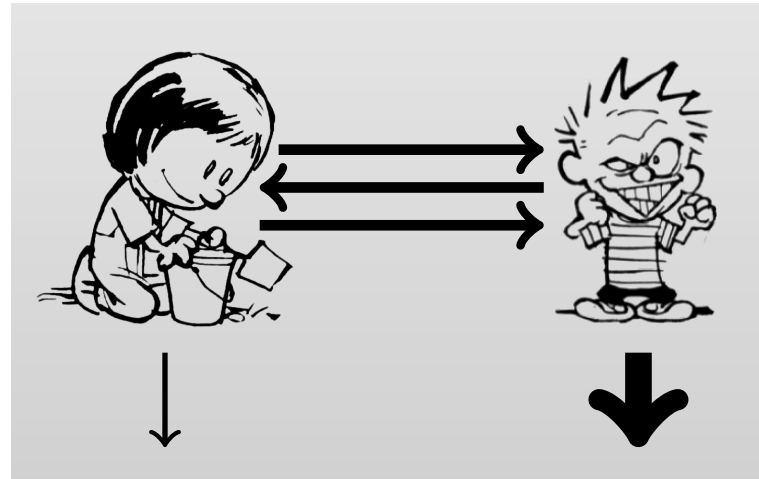
*Security goal: real protocol interaction is **as secure as** the ideal-world interaction*

*For every “attack” against real protocol, there is a way to achieve “same effect” in ideal world*



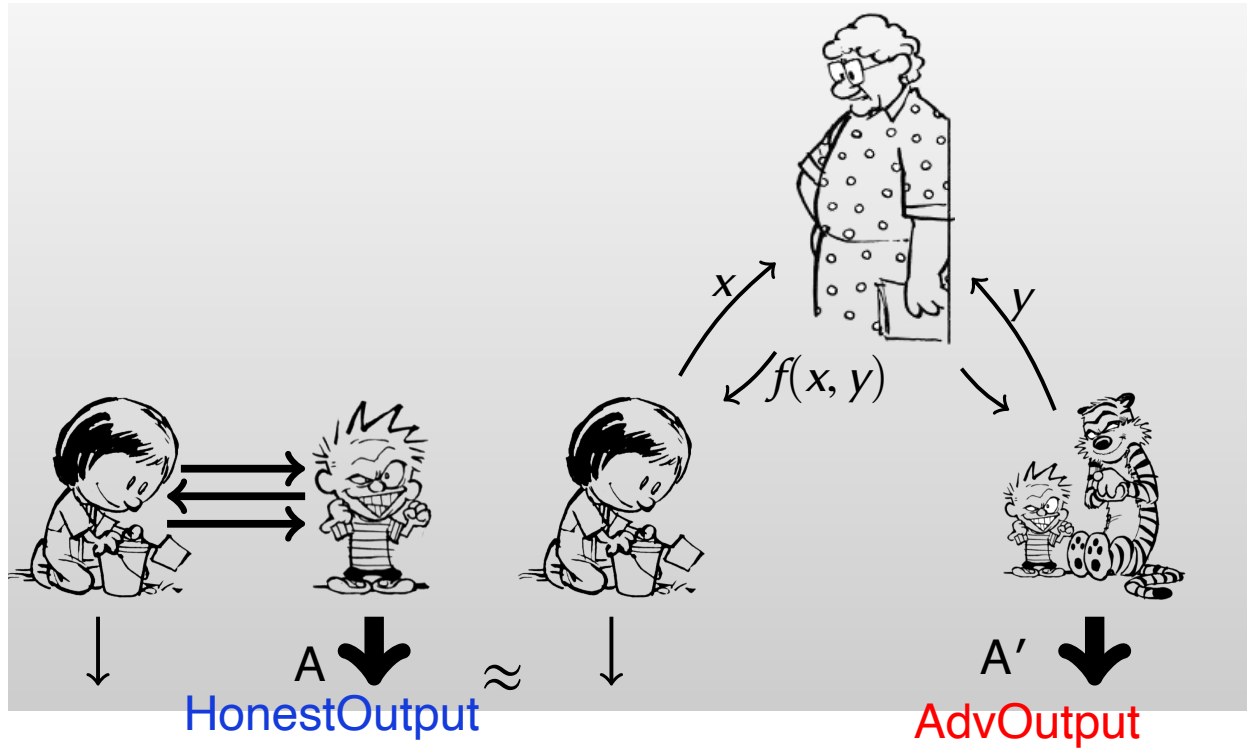
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What is the “effect” of a generic attack?



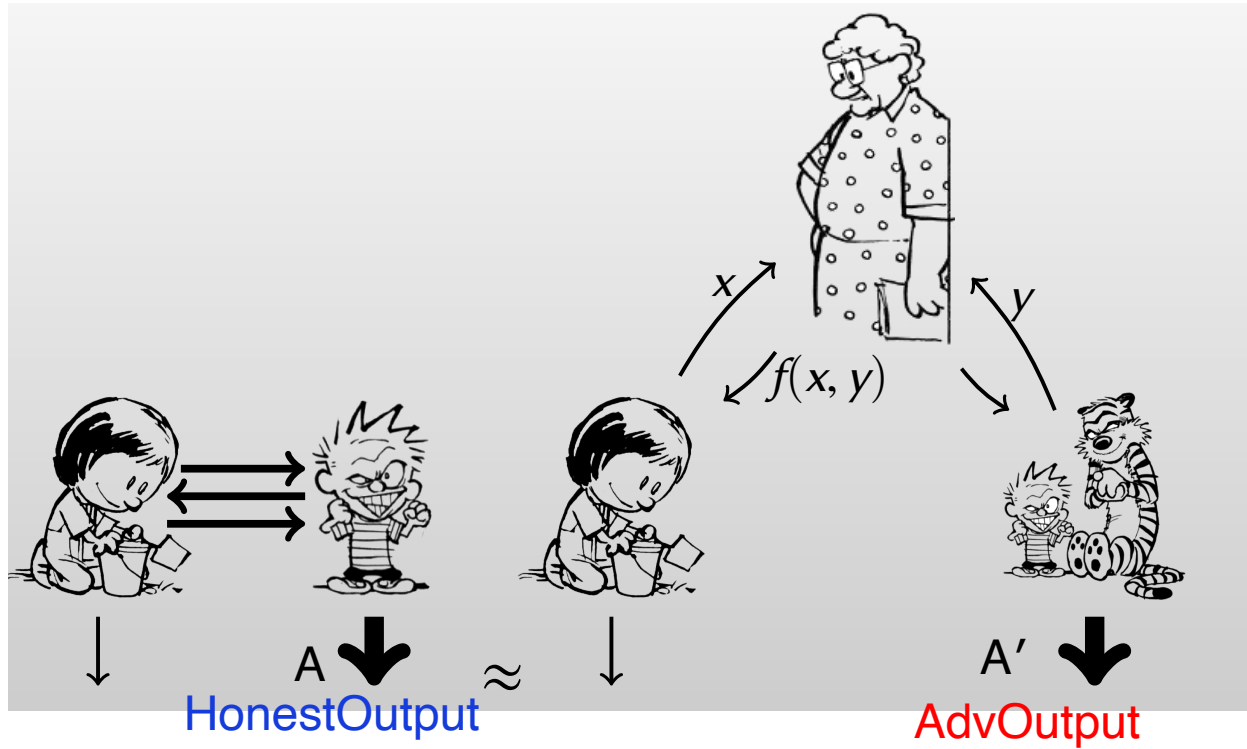
- Something the adversary learns / can compute about honest party
- Some influence on honest party’s output

# Define Security



**Security definition:** For every real-world adversary  $A$ , there exists an ideal adversary  $A'$  s.t. joint distribution (**HonestOutput**, **AdvOutput**) is indistinguishable

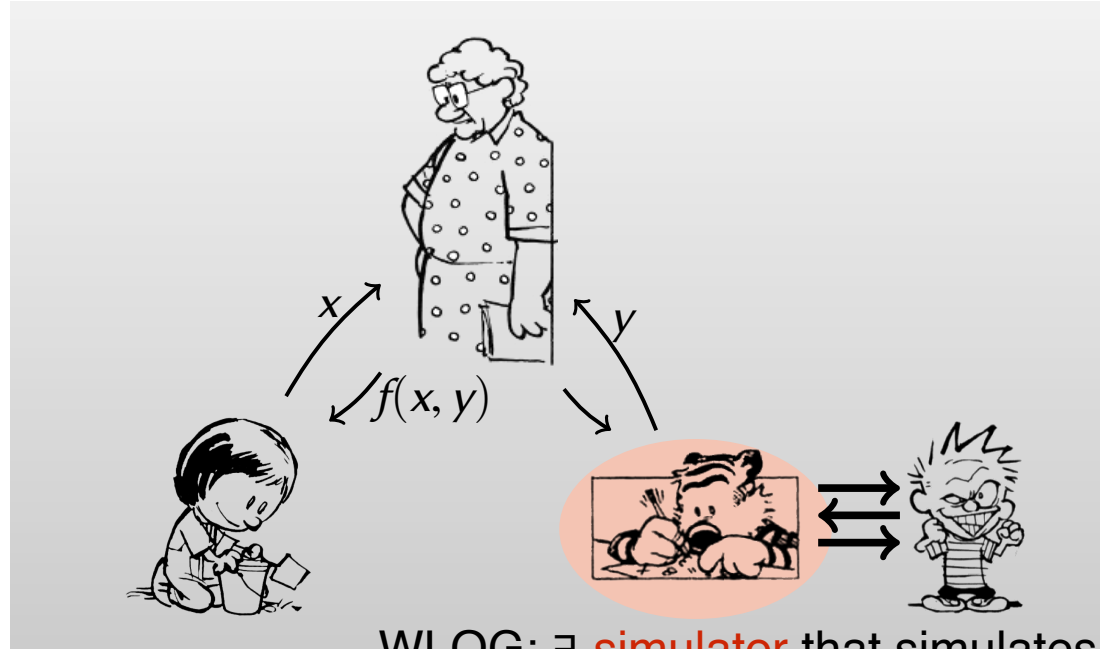
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WLOG:  $\exists$  **simulator** that simulates real-world interaction in ideal world

# Define Security



WLOG:  $\exists$  **simulator** that simulates real-world interaction in ideal world

## Rule of Simulator

1. Send protocol messages that look like they came from honest party
  - Demonstrates that honest party's messages leak no more than  $f(x, y)$
2. **Extract** an  $f$ -input by examining adversary's protocol message
  - "Explains" the effect on honest party's output in terms of ideal world

# Modeling of adversary

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- Adversarial behavior
  - Semi-honest: follows the protocol specification
    - Tries to learn more than allowed by inspecting transcript
  - Malicious: follows any arbitrary strategy
- Adversarial power
  - Polynomial-time
  - Computationally unbounded: information-theoretic security

# Function: Yao's Millionaires' Problem

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$$F(x, y) = \begin{cases} (0, 1), & x < y \\ (1, 0), & x \geq y \end{cases}$$

# Function: Zero-knowledge proof (or SIGMA protocol)

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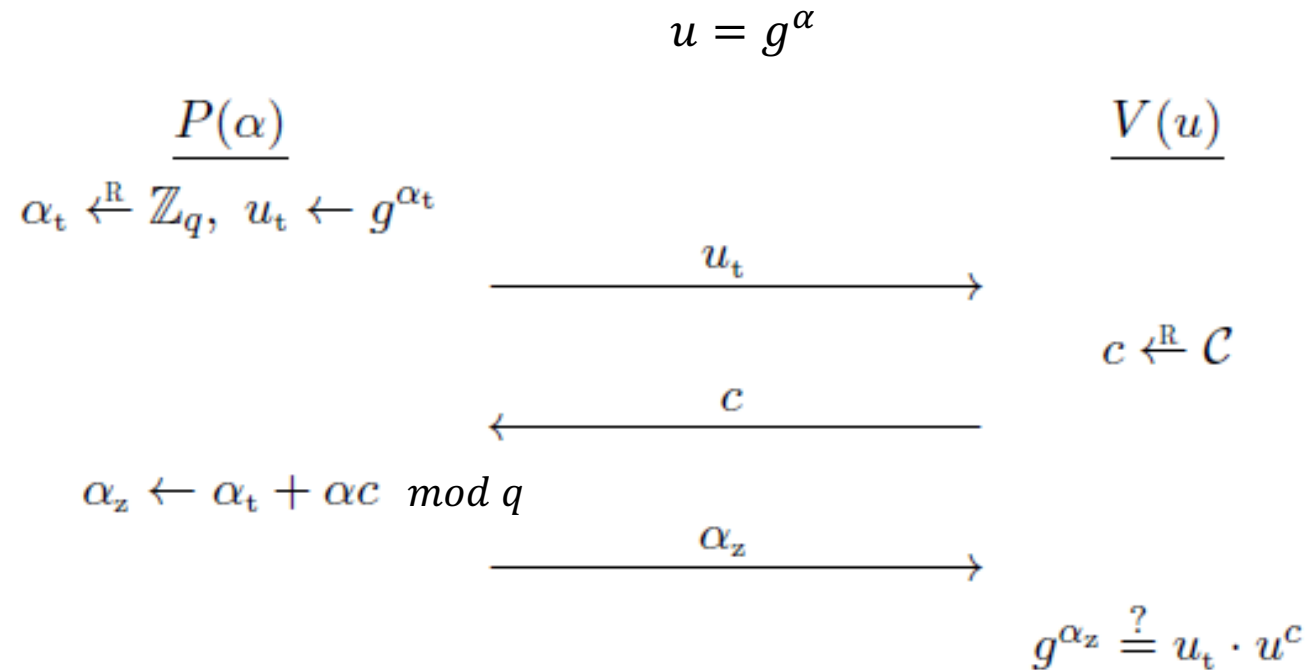
A NP language  $L := \{y \mid \exists x, s. t. (x, y) \in R\}$       Corresponding Relation  $R$

- Prover with input  $(x, y)$  wants to prove that it knows  $x$  such that  $y \in L$

$$F((y, x), y) = (-, b), b = 1 \text{ if } (x, y) \in R$$

Why do we say SIGAMA is an honest verifier zero-knowledge?

# Schnorr Identification



- **Correctness(Completeness):** If P and V execute the protocol honestly, the proof is accepted.

If Prover is an adversary

**Proof-of-knowledge:** If the proof is accepted, we can extract the witness (discrete log)  $\alpha$

- **Honest verifier zero-knowledge** says that: *without knowing* the witness (discrete logarithm), we can generate (simulate) the valid transaction efficiently

If Verifier is a **Honest** adversary



# Function: Zero-knowledge proof (or SIGMA protocol)

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- Prover with input  $(x, y)$  wants to prove that it knows  $x$  such that  $y \in L$

$$F((y, x), y) = (-, b), b = 1 \text{ if } (x, y) \in R$$

- If Prover is the adversary: View is simple; we can extract ‘Prover’s input according to **Soundness (proof-of-knowledge)**:
- If verifier is a Honest adversary: we can simulate the view according to **Honest verifier zero-knowledge**

# Basic tool: Oblivious Transfer (OT)

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Sender S

receiver R



It is theoretically equivalent to MPC as shown by Kilian (1988):

- Given OT, one can build MPC without any additional assumptions
- Similarly, one can directly obtain OT from MPC

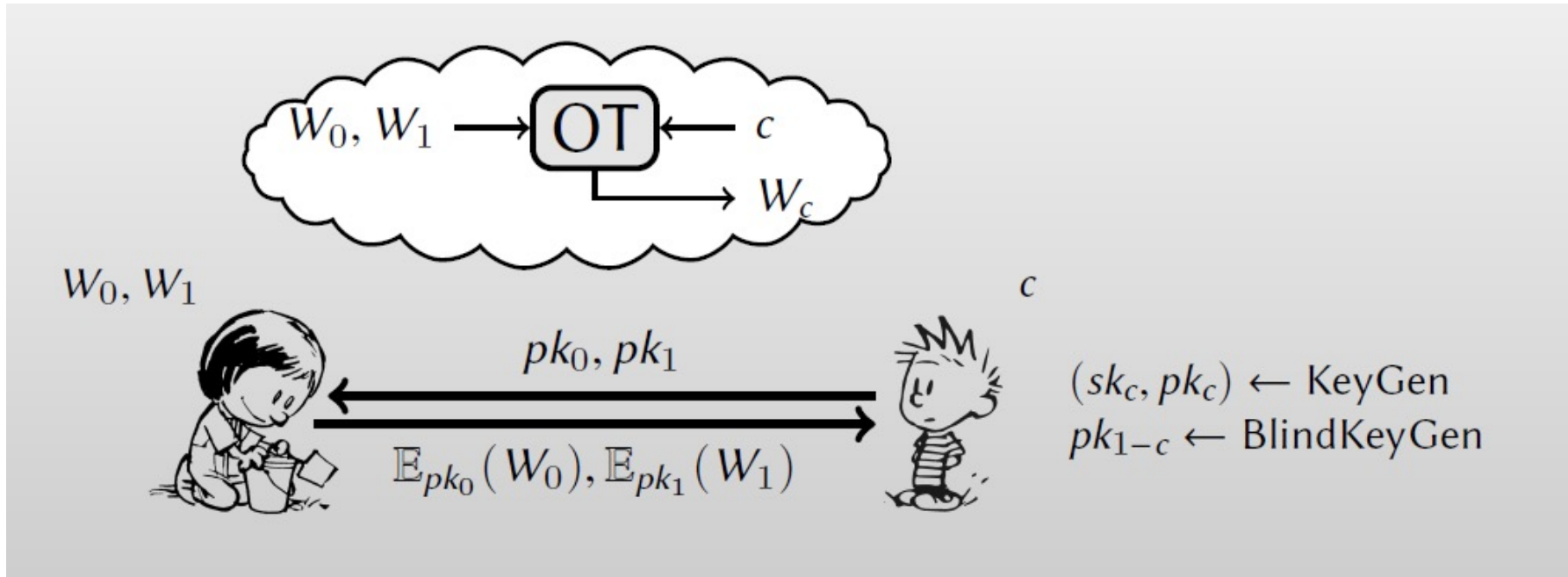
# Oblivious Transfer (OT)

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- The standard definition of 1-out-of-2 OT involves two parties, a Sender  $S$  holding two secrets  $m_0, m_1$ , and a receiver  $R$  holding a choice bit  $b \in \{0, 1\}$
- OT is a protocol allowing  $R$  to obtain  $m_b$  while learning nothing about the "other" secret  $m_{1-b}$
- At the same time,  $S$  does not learn anything at all

# How to construct OT?

- Semi-honest



Need public-key encryption that supports **blind key generation**:

- sample a public key without knowledge of the secret key
- E.g.: ElGamal

# Function for OT

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- A 1-out-of-2 OT is a cryptographic protocol securely implementing the functionality  $F^{OT}$  defined below:
- Parameters:
  - Two parties: Sender S and Receiver R.
  - S has input secrets  $m_0, m_1$  and R has a selection bit  $b \in \{0, 1\}$

Functionality  $F^{OT}$  :

S sends  $m_0, m_1$  to  $F^{OT}$ , and R sends  $b$  to  $F^{OT}$

R receives  $m_b$ , and S receives  $\perp$

# Timetable: MPC

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**Diffie**



**Rivest**



**Rivest**



**Yao**



**Goldwasser**



**Hellman**



**Shamir**



**Adelman**



**Adelman**



**Dertouzos**



**Micali**



**Rackoff**

---

**1976**

**New directions**

**1977**

**RSA**

**1978**

**Homomorphic Enc**

**1982**

**MPC**

**1985**

**Zero Knowledge**

# History of MPC

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- The idea of secure computation was introduced by Andrew Yao in the early 1980s (Yao, 1982)
- Secure computation was primarily of only theoretical interest for the next twenty years
- In the early 2000s, algorithmic improvements and computing costs make it more realistic to build practical systems, e.g. Fairplay (Malkhi et al., 2004)
- Since then, the speed of MPC protocols has improved by more than five orders of magnitude

# Our step

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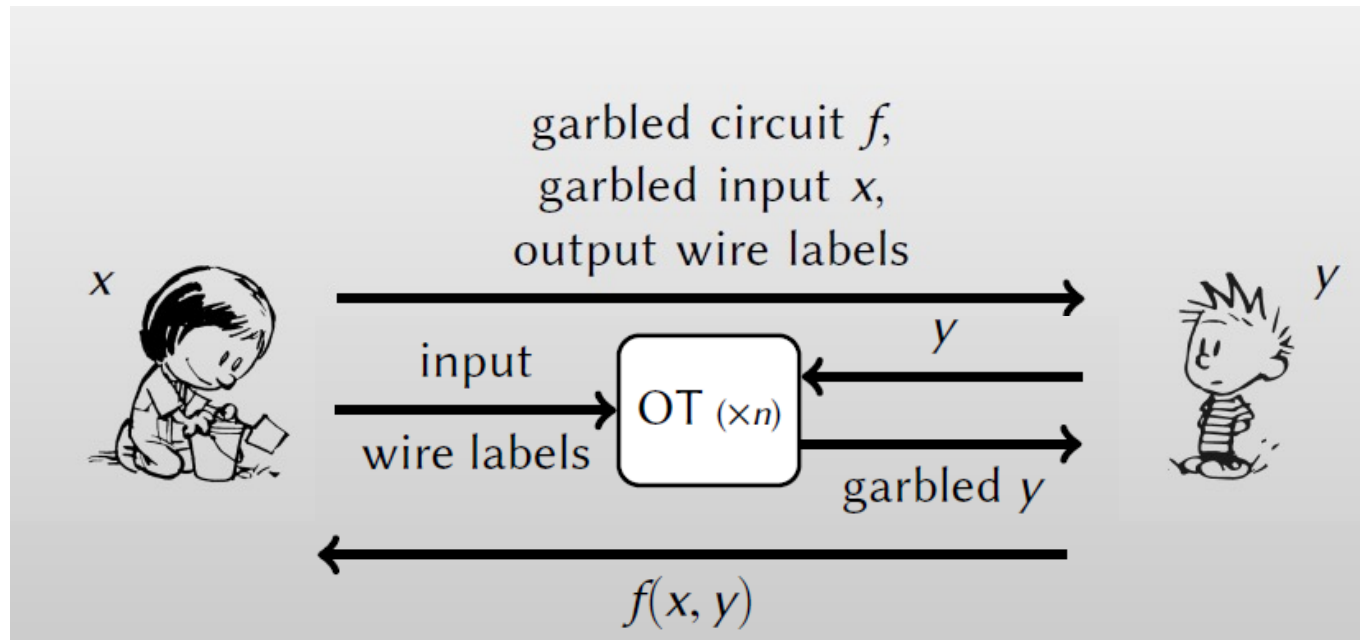
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- 3 Custom protocol:** private set intersection



# First: Two-party computation

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- Every computation of function could be transferred to **computing a Boolean circuit**.
- Yao's protocol: semi-honest secure (2-party) computation for Boolean circuits



Before we start, , so we focus on semi-honest case

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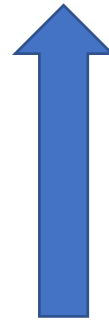
**Malicious secure MPC for *any circuit***

**GMW compiler**

**[GMW87]**

Commitment

Zero-knowledge proof



**Semi-honest secure MPC for *any circuit***

Goldreich-Micali-Wigderson (GMW)

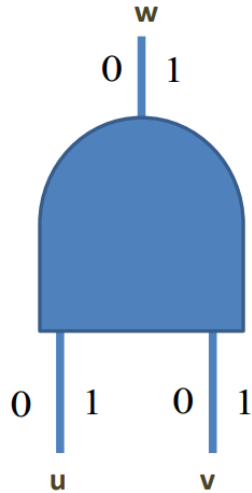
Yao etc.

[GMW87] Goldreich, O., S. Micali, and A. Wigderson. 1987. "How to Play any Mental Game or A Completeness Theorem for Protocols with Honest Majority".

# Yao's Garble Circuit (two-party, Boolean)

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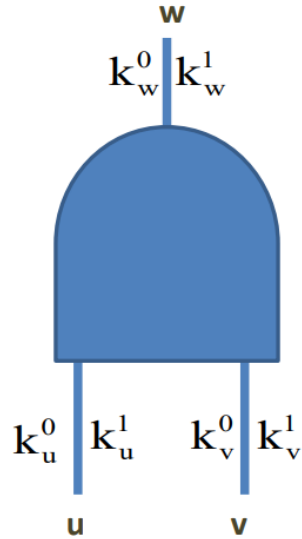
- Take AND gate for example
- $F(u, v) = (w, w)$



| u | v | w |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

# Yao's Garble Circuit (two-party, Boolean)

- $F(u, v) = (w, w)$

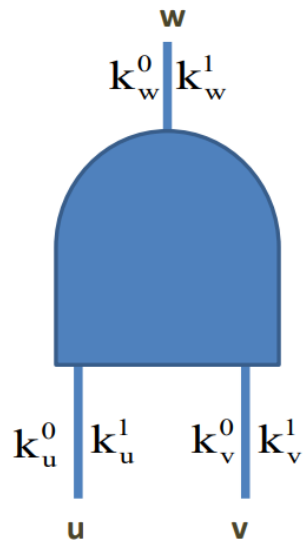


$E_{k_1}(E_{k_2}(m))$  is the double AES enc of  $m$  with  $k_1$  and  $k_2$

| u       | v       | w                             |
|---------|---------|-------------------------------|
| $k_u^0$ | $k_v^0$ | $E_{k_u^0}(E_{k_v^0}(k_w^0))$ |
| $k_u^0$ | $k_v^1$ | $E_{k_u^0}(E_{k_v^1}(k_w^0))$ |
| $k_u^1$ | $k_v^0$ | $E_{k_u^1}(E_{k_v^0}(k_w^0))$ |
| $k_u^1$ | $k_v^1$ | $E_{k_u^1}(E_{k_v^1}(k_w^1))$ |

- U sends all the ciphertexts  $E_{k'}(E_{k''}(k'''))$  in volume  $w$  to V
- U sends  $k_u^u$  to V
- U sends  $k_w^0, k_w^1$  to V

# Yao's Garble Circuit (two-party, Boolean)

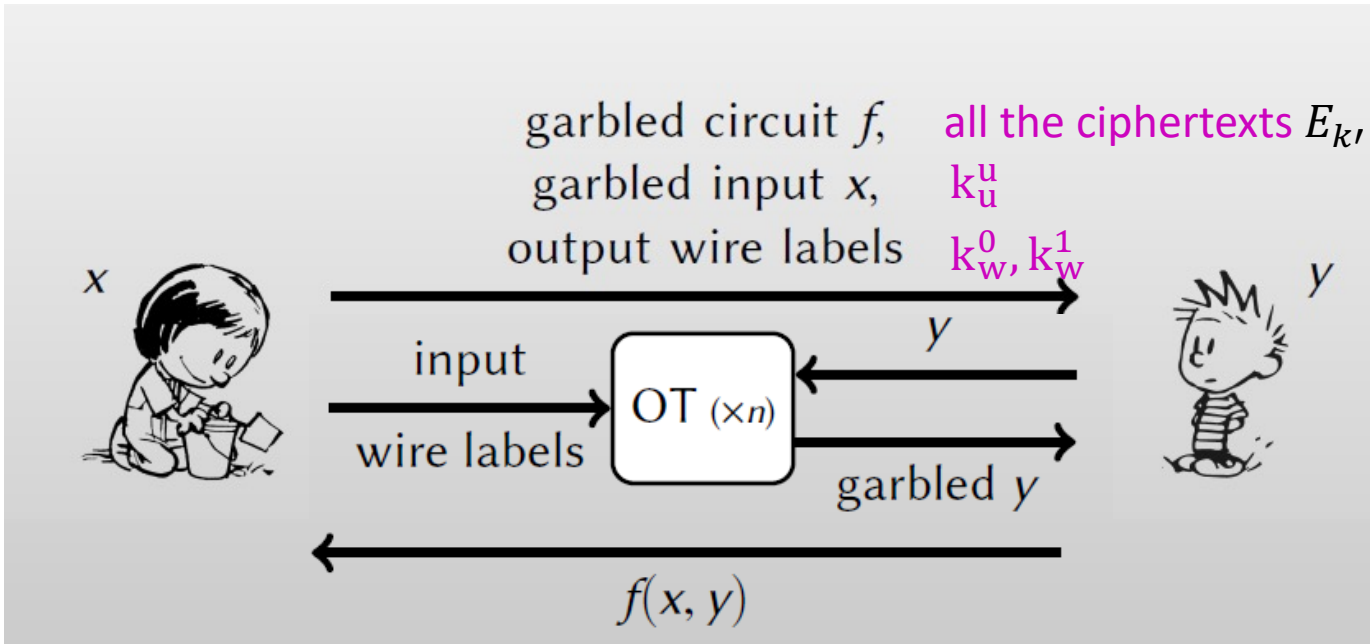


| u       | v       | w                             |
|---------|---------|-------------------------------|
| $k_u^0$ | $k_v^0$ | $E_{k_u^0}(E_{k_v^0}(k_w^0))$ |
| $k_u^0$ | $k_v^1$ | $E_{k_u^0}(E_{k_v^1}(k_w^0))$ |
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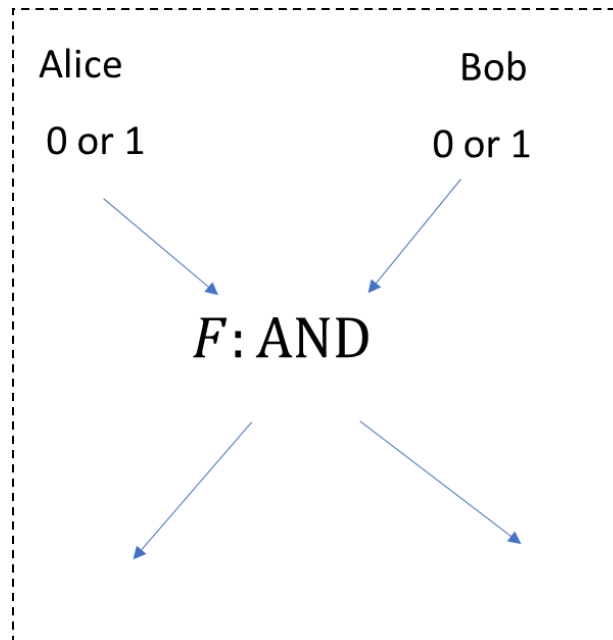
- all the ciphertexts  $E_{k^i}(E_{k^{j'}}(k^{l''}))$  in volume w,
- $k_u^u$
- $k_w^0, k_w^1$

With  $k_u^u$  and  $k_v^v$ , V can decrypt  $k_w^w$



# A fun application

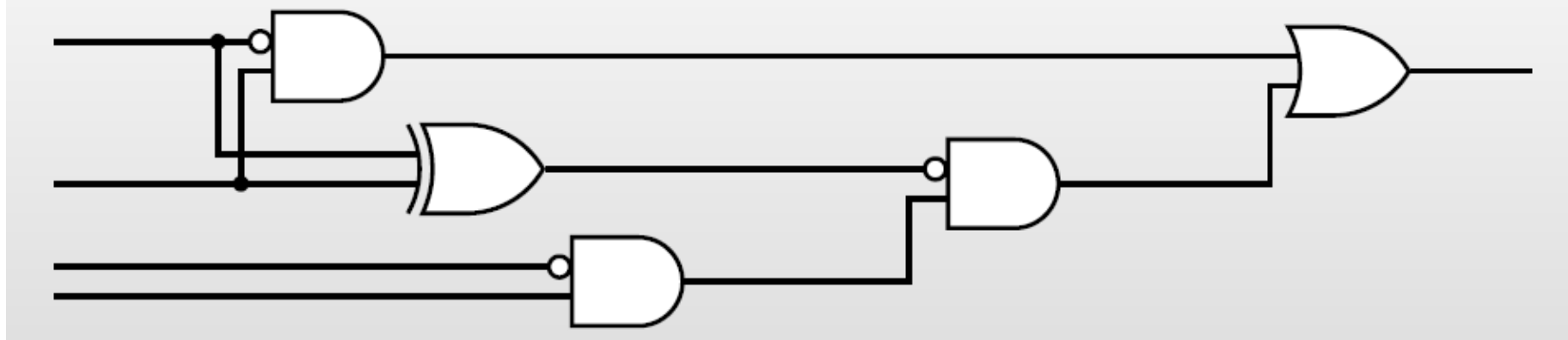
- Bob and Alice want to check if they are interested in dating
  - If both are yes, the output is yes
  - If one is no, the output is no



<Pride and Prejudice>

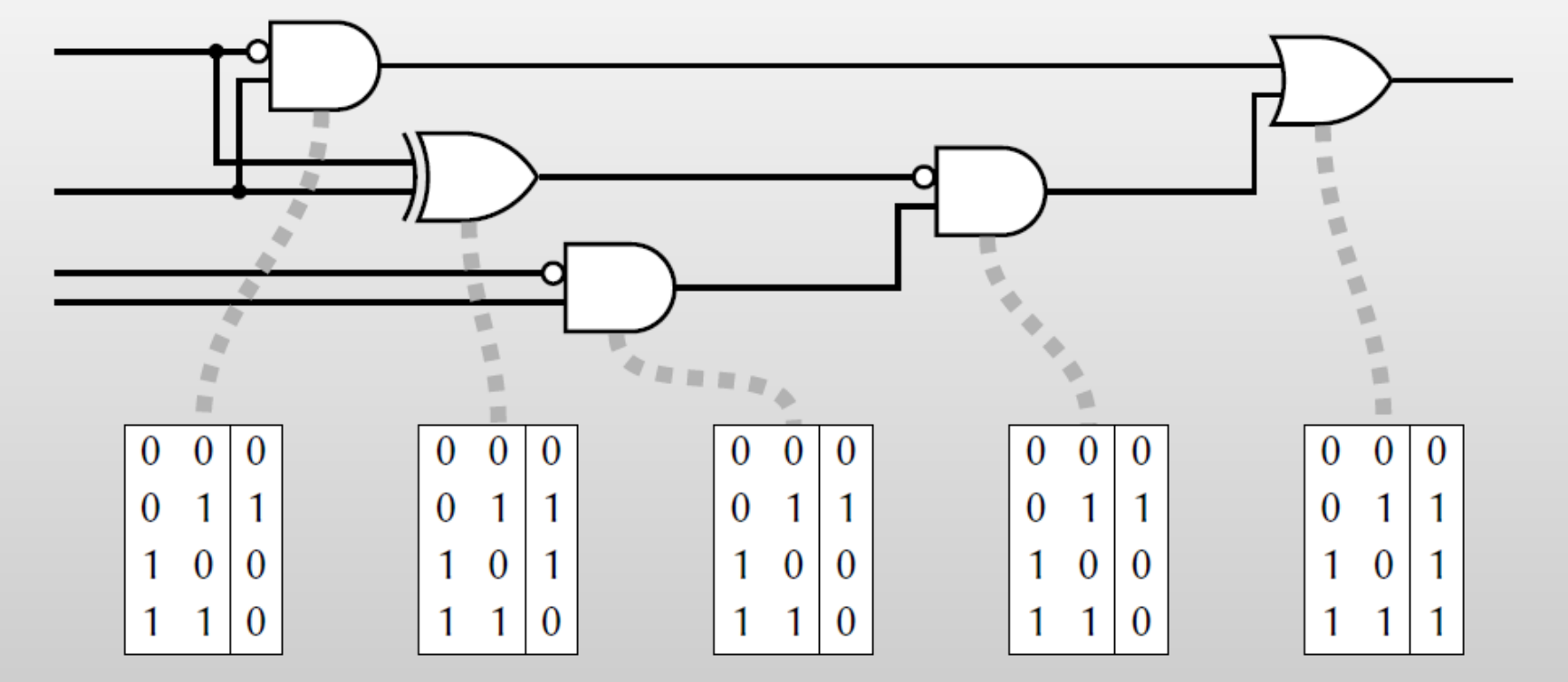
# Garbled general circuit framework

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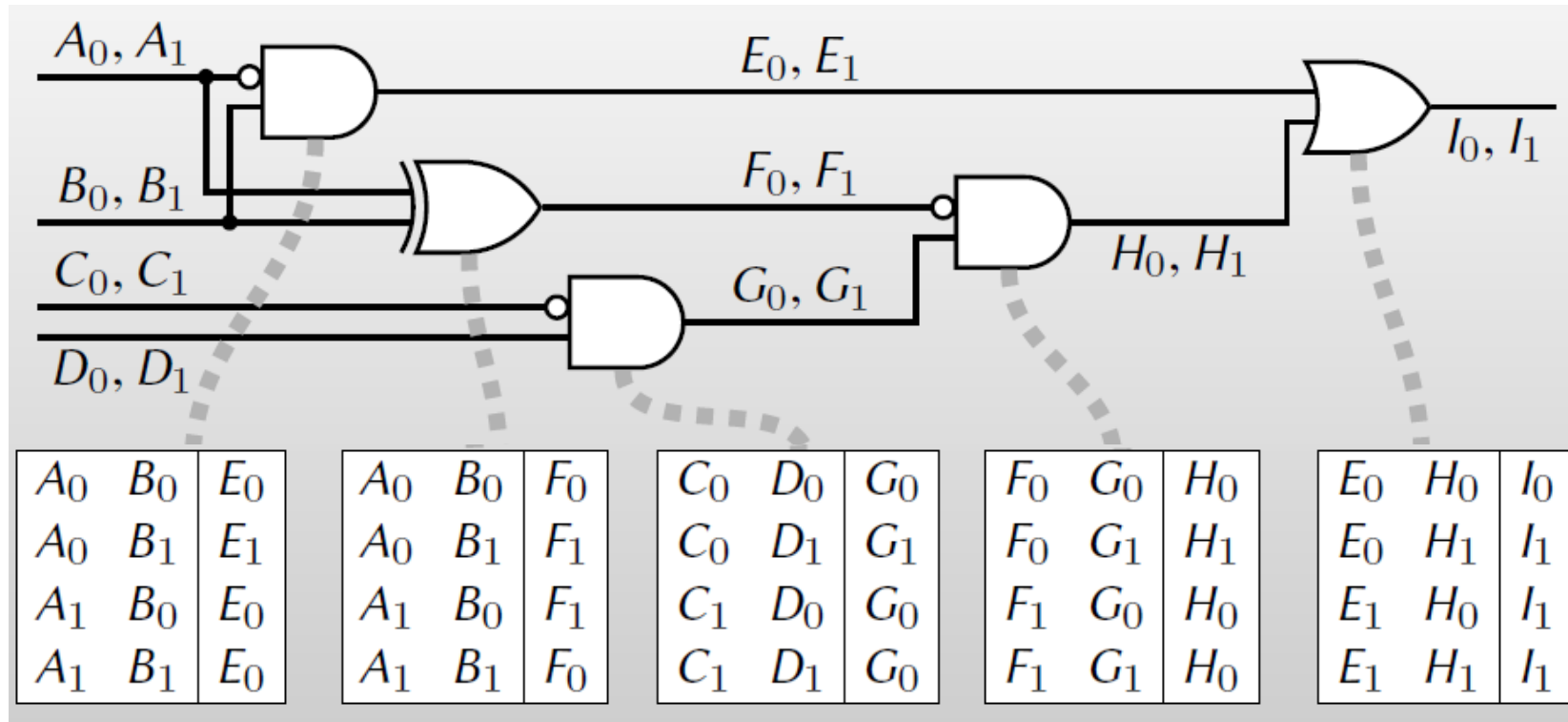


# Garbled general circuit framework



Garbling a circuit:

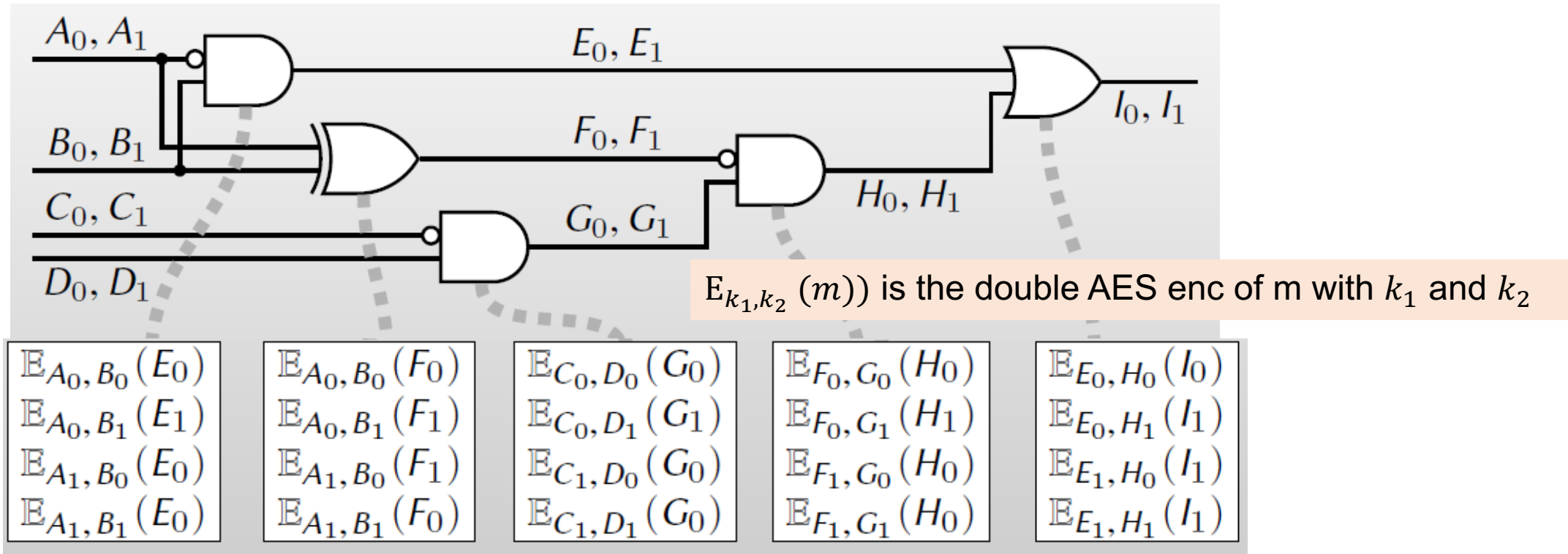
# Garbled general circuit framework



## Garbling a circuit:

- Pick random **labels**  $W_0; W_1$  on each wire

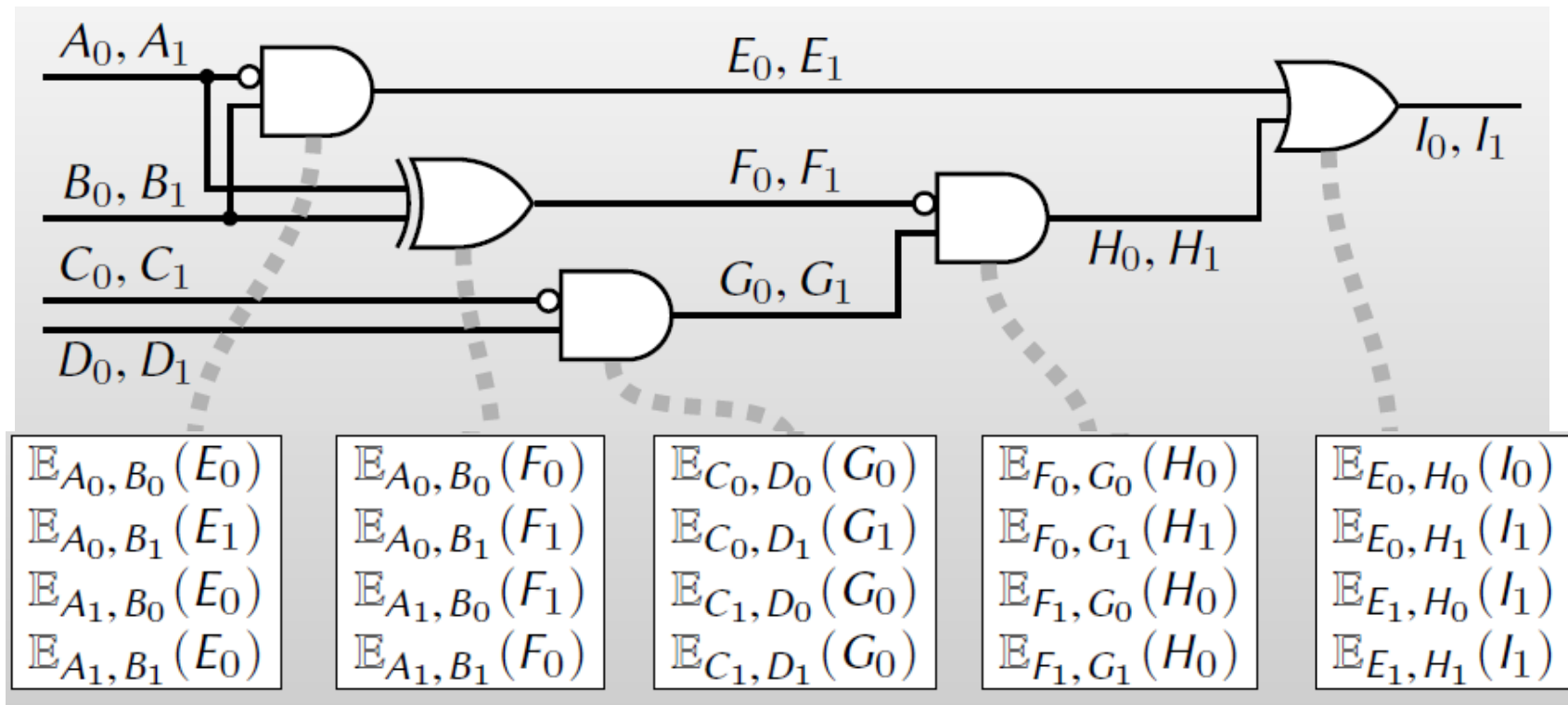
# Garbled general circuit framework



## Garbling a circuit:

- Pick random **labels**  $W_0; W_1$  on each wire
- “Encrypt” truth table of each gate

# Garbled general circuit framework

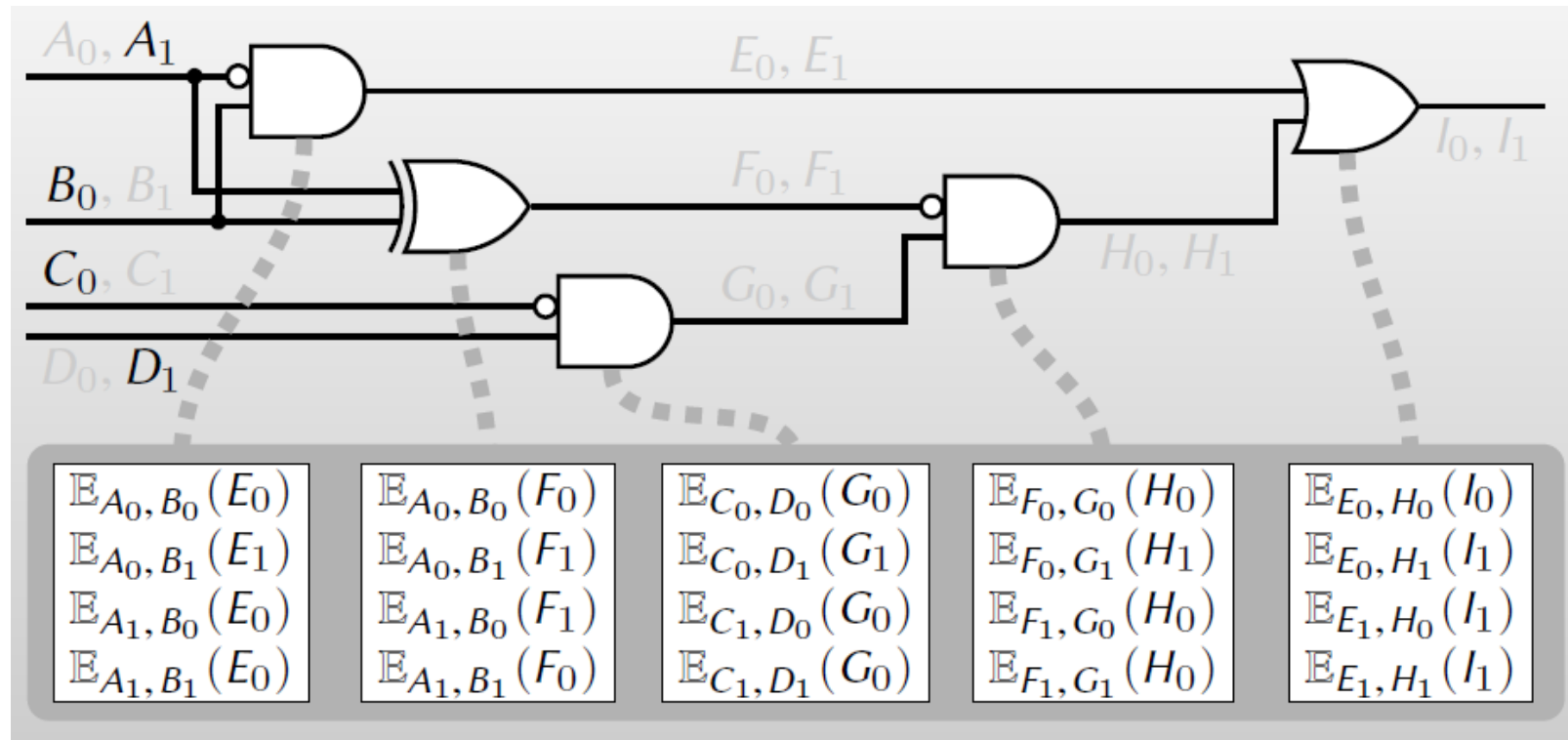


## Garbling a circuit:

- Pick random **labels**  $W_0; W_1$  on each wire
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## Garbled evaluation:

# Garbled general circuit framework

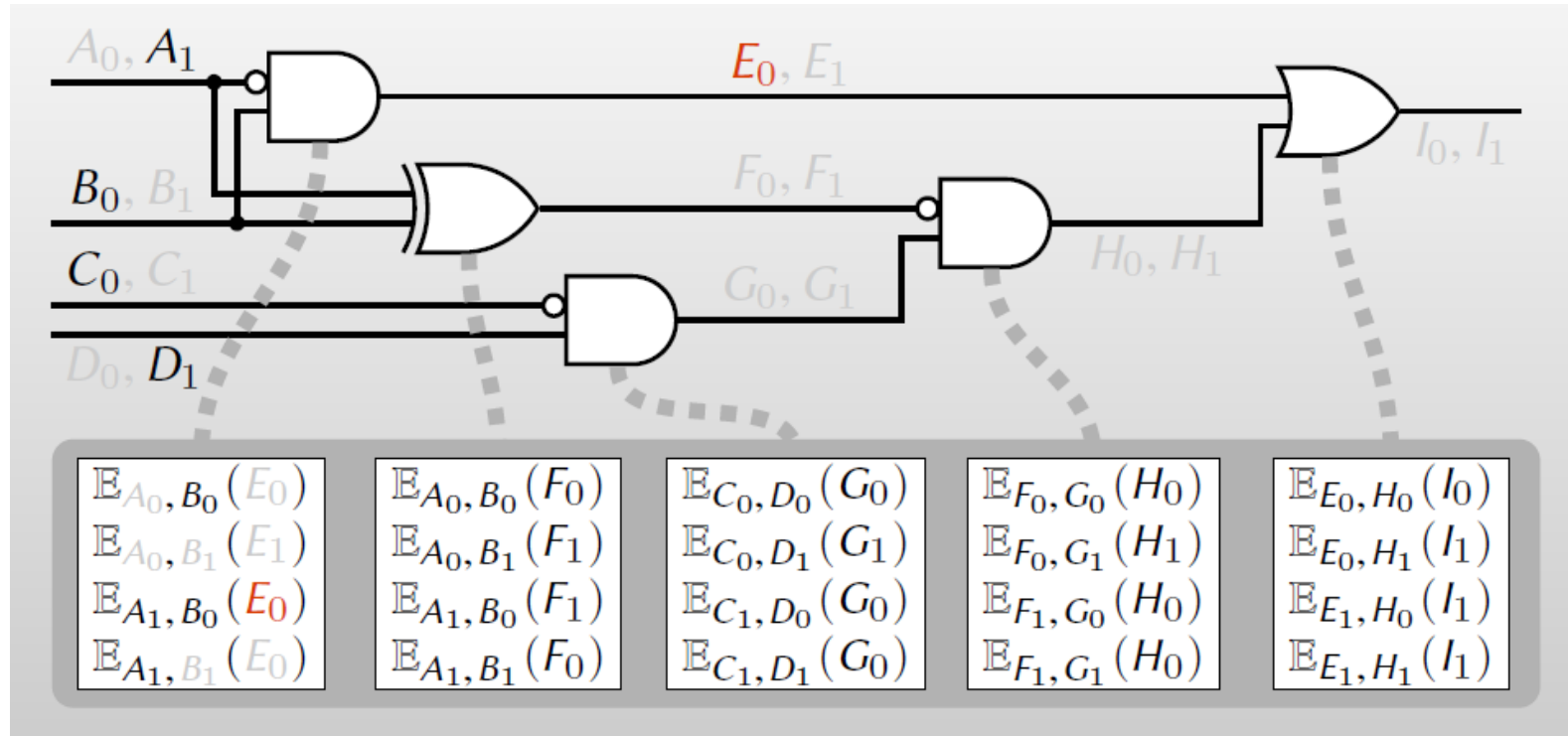


## Garbling a circuit:

- Pick random **labels**  $W_0; W_1$  on each wire
- “Encrypt” truth table of each gate
- **Garbled circuit** all encrypted gates
- **Garbled encoding** one label per wire

## Garbled evaluation:

# Garbled general circuit framework



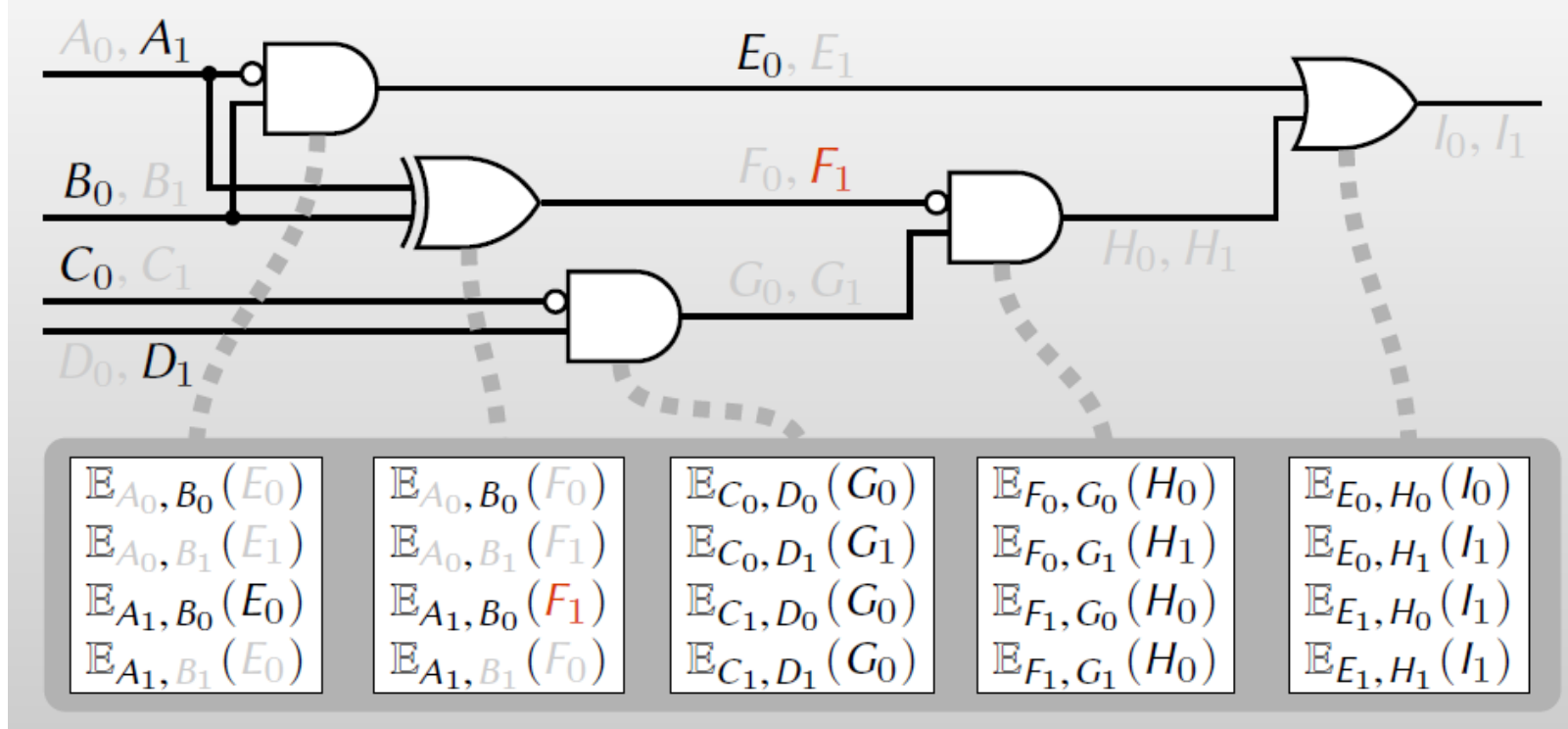
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## Garbled evaluation:

- Only one ciphertext per gate is decryptable

# Garbled general circuit framework



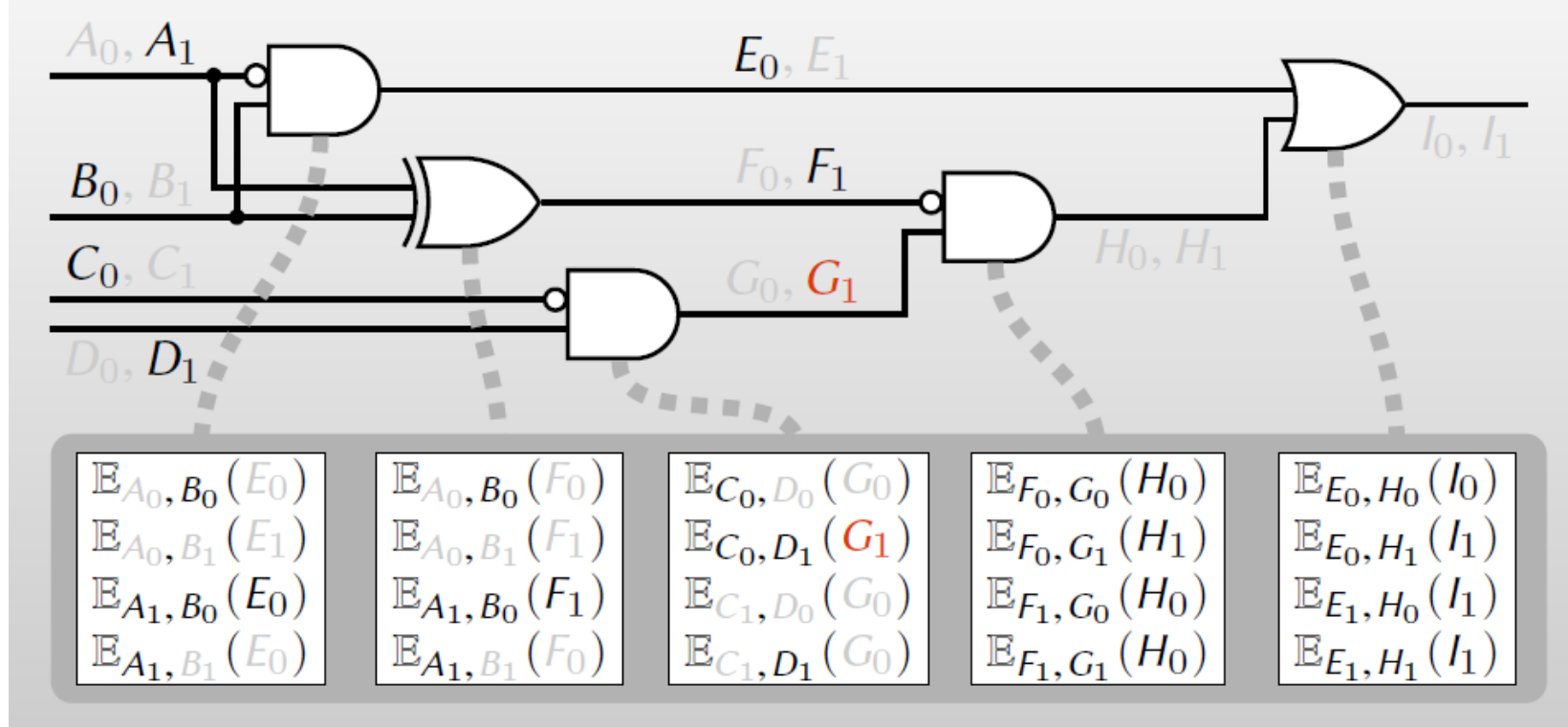
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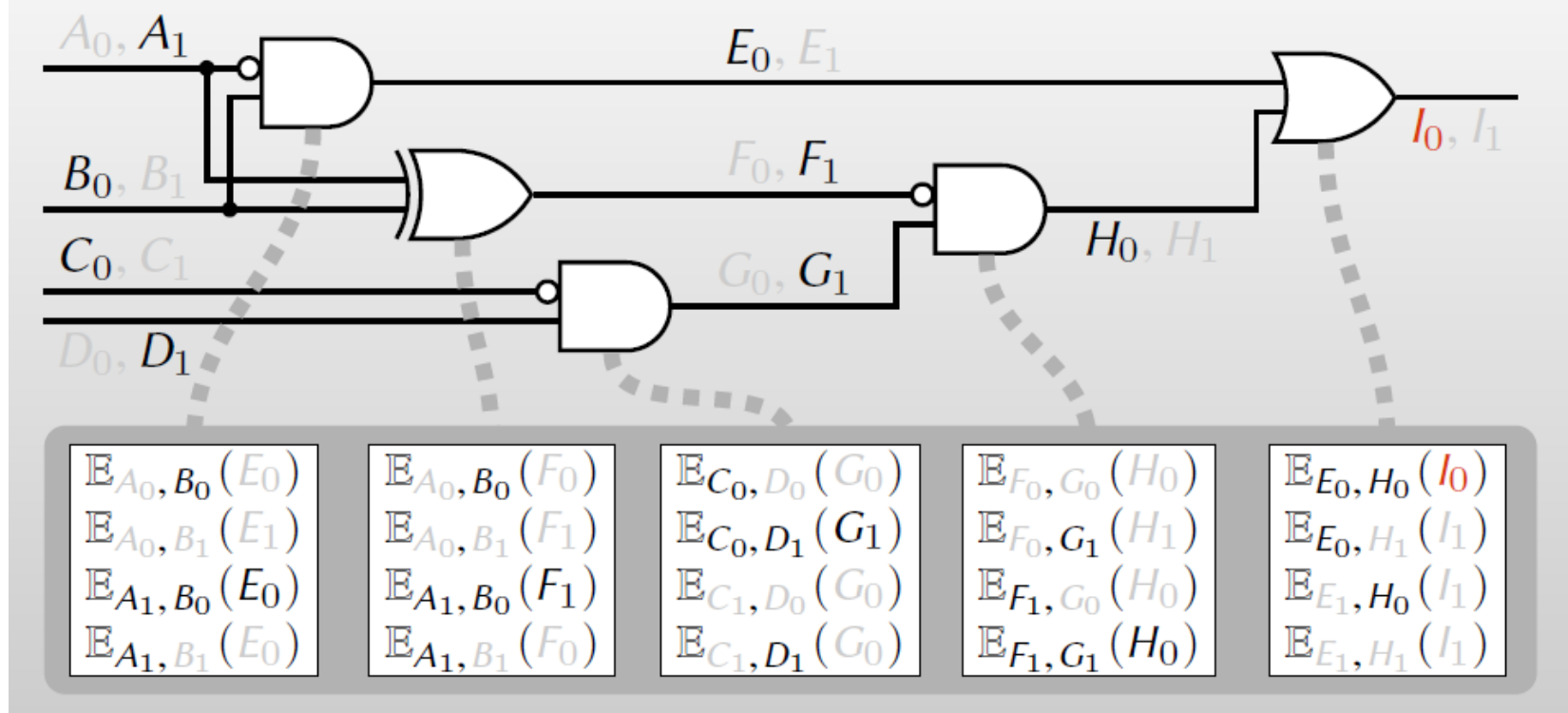
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# Garbled general circuit framework



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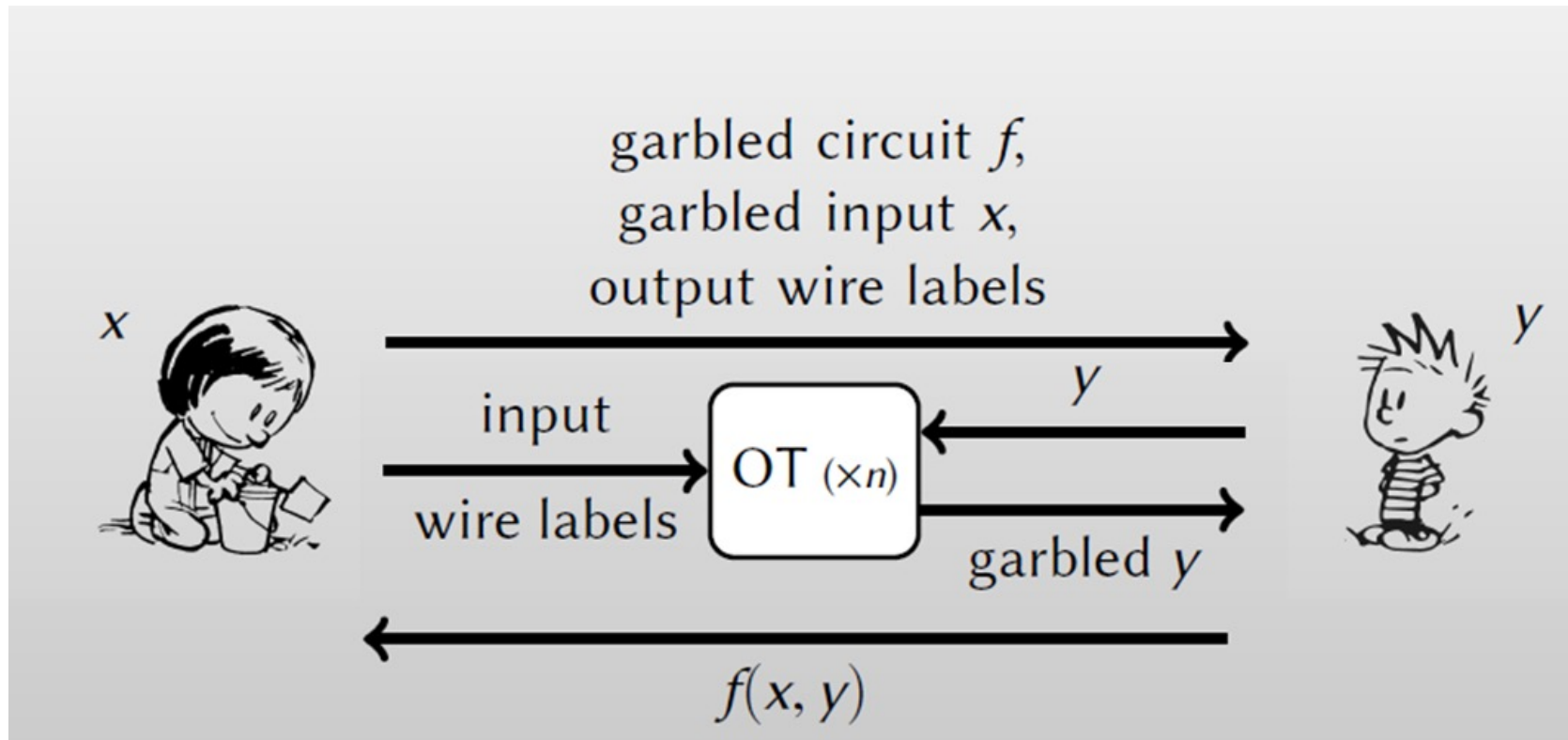
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- “Encrypt” truth table of each gate
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- **Garbled encoding** one label per wire

## Garbled evaluation:

- Only one ciphertext per gate is decryptable
- Result of decryption = value on outgoing wire

Security

# Yao's Protocol



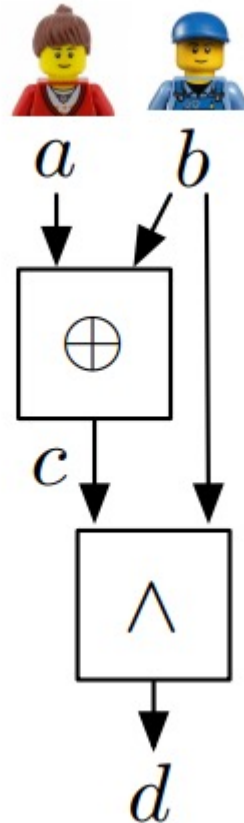
- Two party
- For a **Boolean circuit**.

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How about Multi-party and arithmetic / Boolean circuit?

# GMW (multiparty, Boolean)

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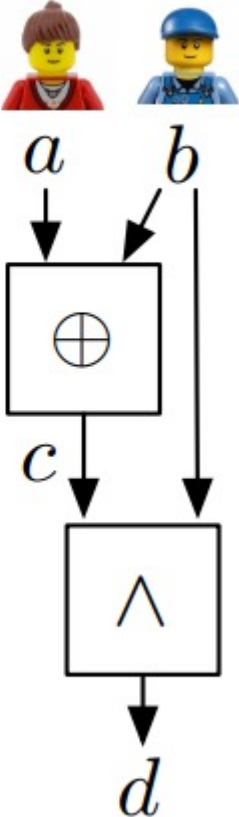
[GMW87] Goldreich, O., S. Micali, and A. Wigderson. 1987. "How to Play any Mental Game or A Completeness Theorem for Protocols with Honest Majority".

# GMW (multiparty, Boolean)

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Secret share inputs:

|   |          |   |
|---|----------|---|
|  |          |  |
| $a = a_1$   | $\oplus$ | $a_2$   |
| $b = b_1$   | $\oplus$ | $b_2$   |



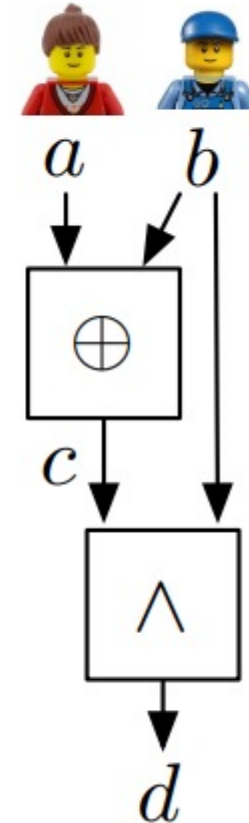
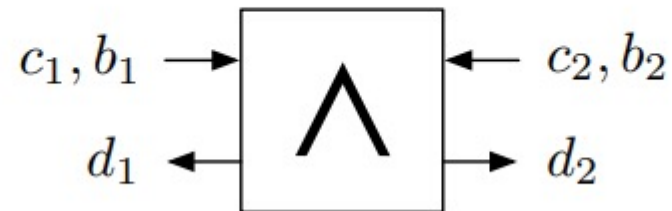
# GMW (multiparty, Boolean)

Secret share inputs:


$$a = a_1 \oplus a_2$$
$$b = b_1 \oplus b_2$$

Non-Interactive XOR gates:  $c_1 = a_1 \oplus b_1$ ;  $c_2 = a_2 \oplus b_2$

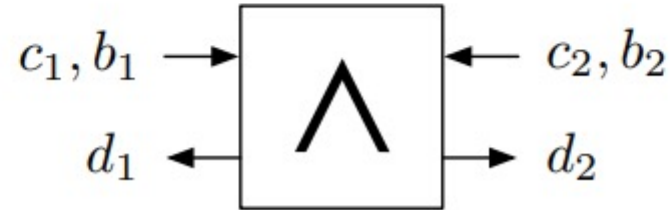
Interactive AND gates:



# GMW (multiparty, Boolean)

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Interactive AND gates:



- One AND gate requires the execution of 1-out-of-4 OT


$$d_2 = (c_1 \oplus c_2)(b_1 \oplus b_2) - d_1$$

$$\begin{aligned} &(c_1 \oplus 0)(b_1 \oplus 0) - d_1, \\ &(c_1 \oplus 0)(b_1 \oplus 1) - d_1, \\ &(c_1 \oplus 1)(b_1 \oplus 0) - d_1, \\ &(c_1 \oplus 1)(b_1 \oplus 1) - d_1 \end{aligned}$$



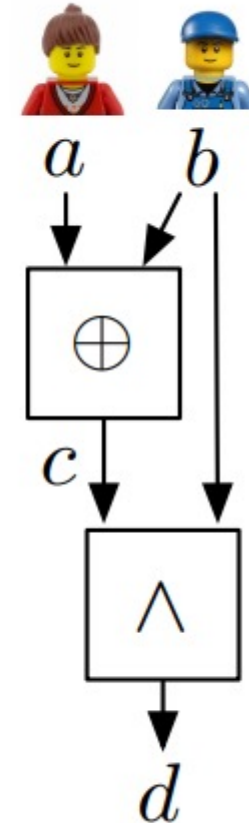
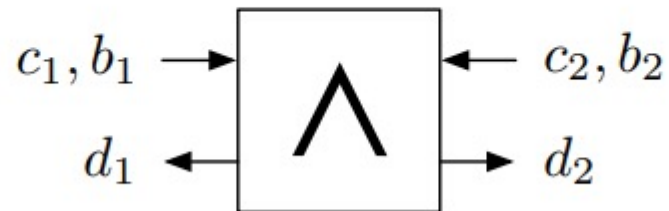
# GMW (multiparty, Arithmetic/Boolean)

Secret share inputs:


$$a = a_1 \oplus a_2$$
$$b = b_1 \oplus b_2$$

Non-Interactive XOR gates:  $c_1 = a_1 \oplus b_1$ ;  $c_2 = a_2 \oplus b_2$

Interactive AND gates:



Not difficult to extend to Multi-party by using 1-out-of-k OT



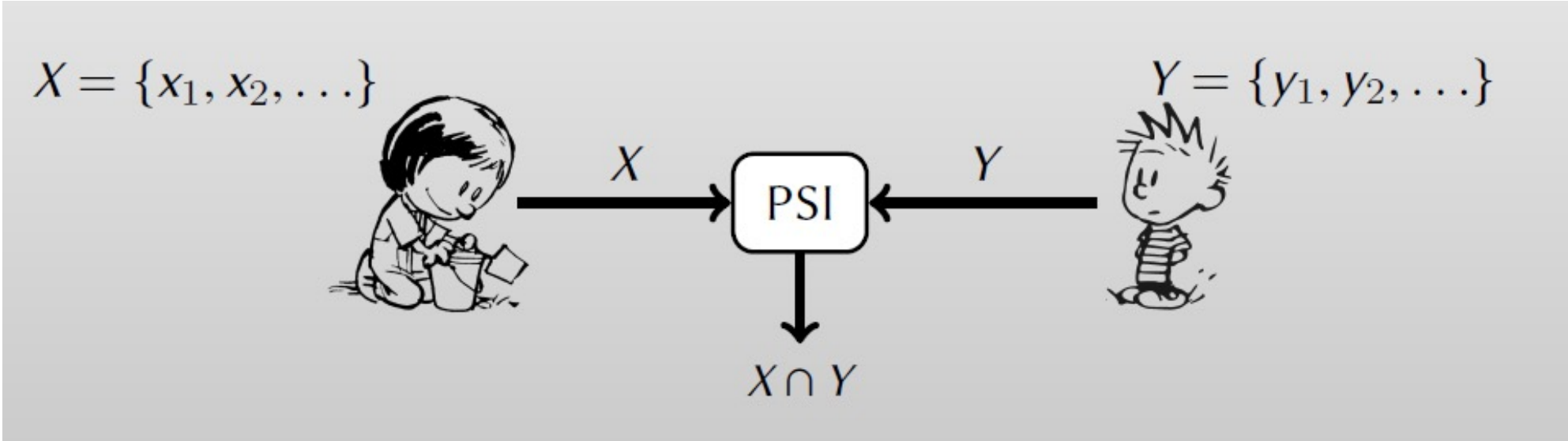
# Our step

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- 1 Secure computation:** Concepts & definitions
- 2 General constructions:** Yao's protocol, and others
- 3 Custom protocol:** private set intersection

# Custom protocol: private set intersection (PSI)

Special case of secure 2-party computation:



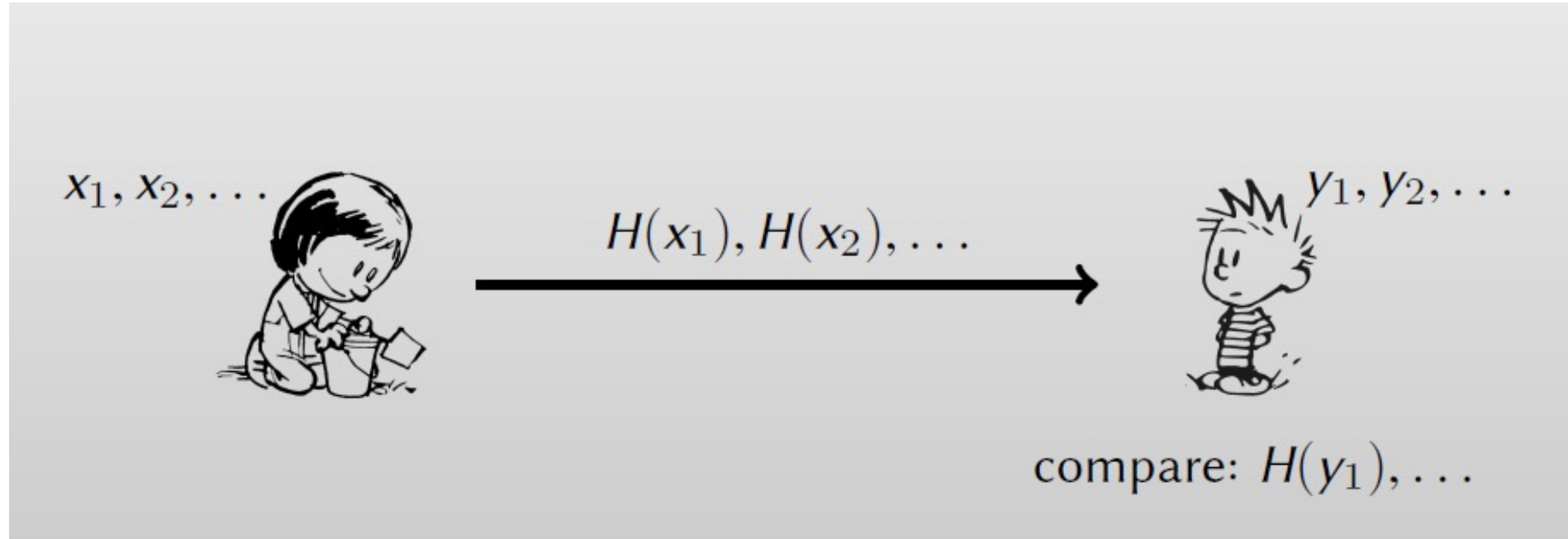
# PSI applications

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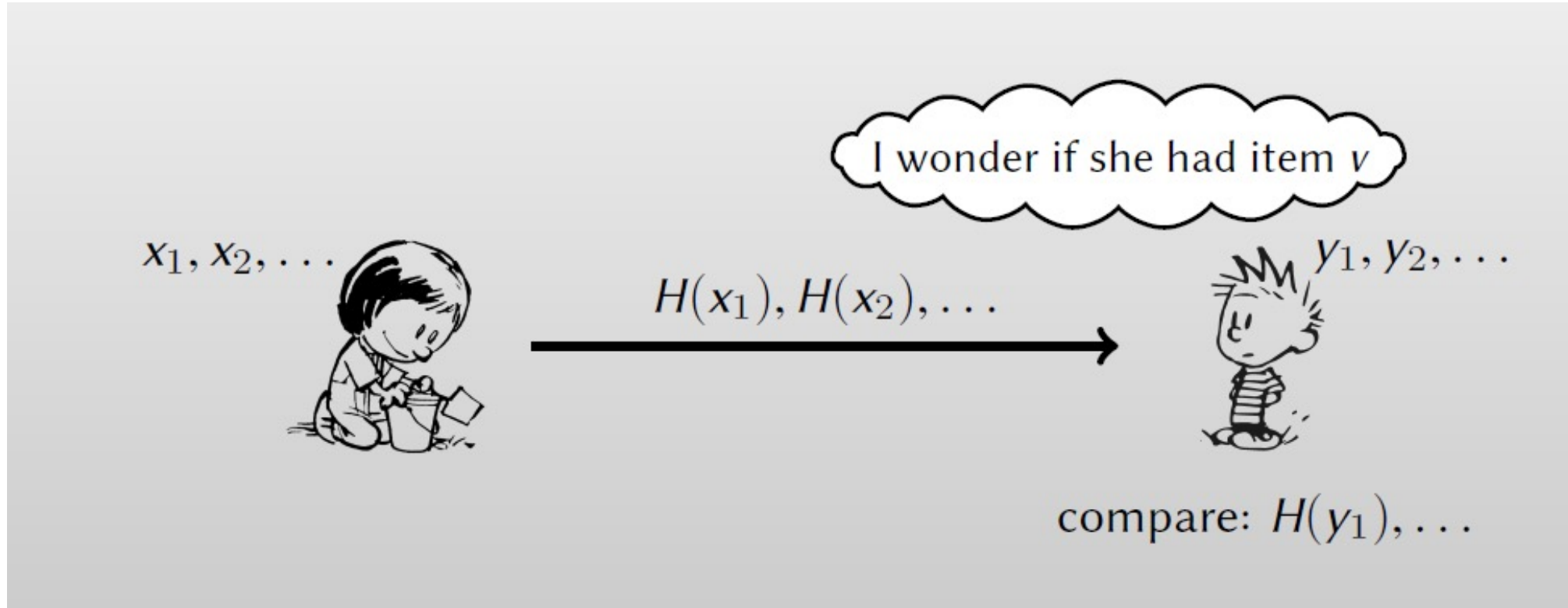
- Contact discovery, when signing up for WhatsApp
  - $X$  = address book in my phone (phone numbers)
  - $Y$  = WhatsApp user database
- Private scheduling
  - $X$  = available timeslots on my calendar
  - $Y$  = available timeslots on your calendar
- Ad conversion rate
  - $X$  = users who saw the advertisement
  - $Y$  = customers who bought the product
- etc

# “Obvious” protocol

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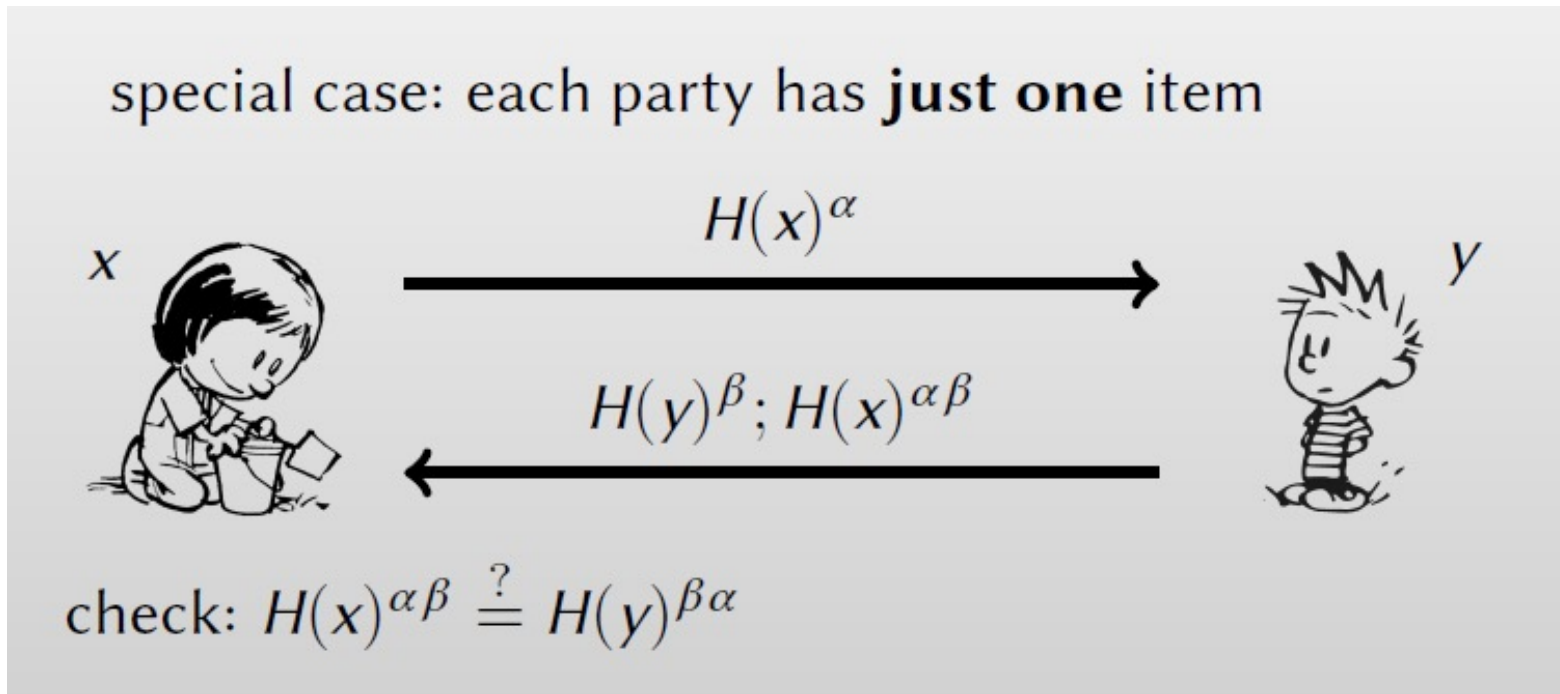


# “Obvious” protocol



- **INSECURE:** Receiver can test any  $v \in \{x_1, x_2, \dots\}$  or not offline
- Problematic if items have low entropy (e.g., phone numbers)

# Classical protocol: Diffie-Hellman



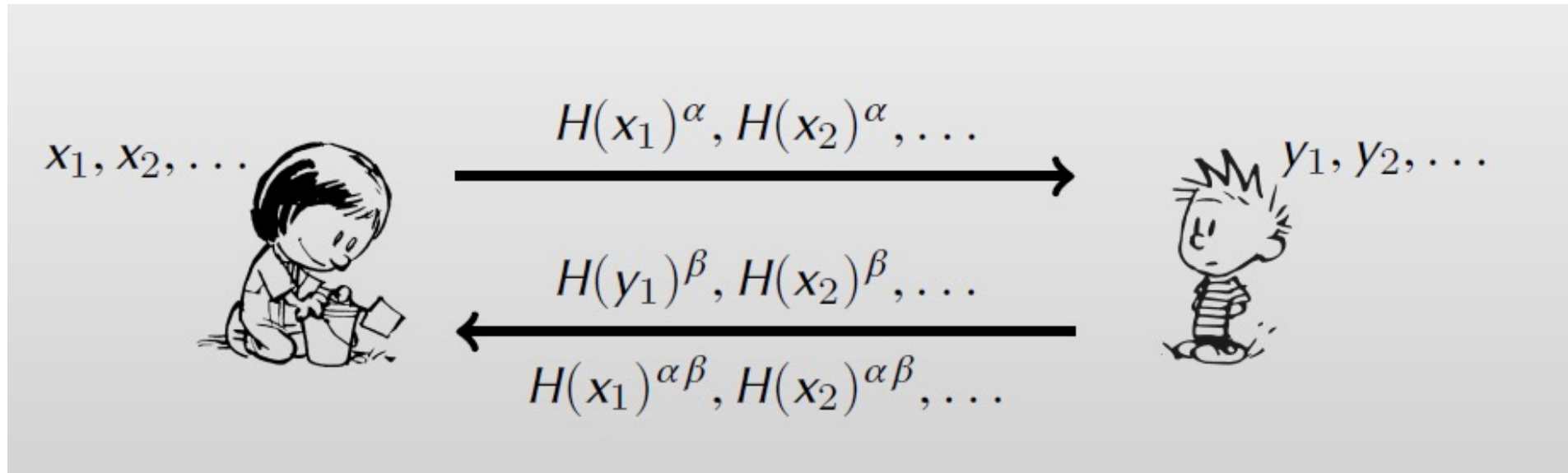
where  $H$  is a hash function with image of a group  $G = \langle g \rangle$

Idea:

- If  $x = y$ ,  $H(x)^{\alpha\beta} = H(y)^{\alpha\beta}$
- If  $x \neq y$ , they are random

# Classical protocol: Diffie-Hellman

Drawback:  $O(n)$  expensive exponentiations



where  $H$  is a hash function with image of a group  $G = \langle g \rangle$

Idea:

- If  $x = y$ ,  $H(x)^{\alpha\beta} = H(y)^{\alpha\beta}$
- If  $x \neq y$ , they are random

## There are other solutions with trade-offs using

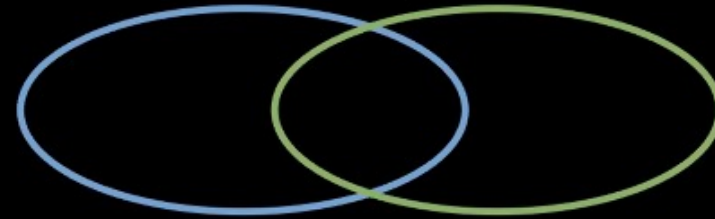
- Yao's protocol
- OT
- Etc.





PSI on **small sets** (hundreds)

- ▶ private availability poll
- ▶ key agreement techniques



PSI on **large sets** (millions)

- ▶ double-registered voters
- ▶ OT extension; combinatorial tricks



PSI on **asymmetric sets** (100 : billion)

- ▶ contact discovery; password checkup
- ▶ offline phase; leakage



**computing on the intersection**

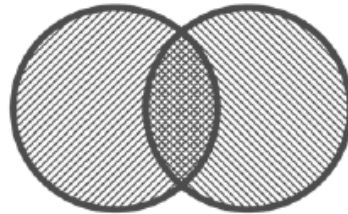
- ▶ sales statistics about intersection
- ▶ generic MPC

# PSI: intersection of leaked password

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Alice91 | 0791    🔒 \*\*\*\*  
A.Sample | 1234    🔒 \*\*\*\*  
Bob | b4dp455    🔒 \*\*\*\*  
Alice | 12345    🔒 \*\*\*\*



🔒 🔒 ? \*  
🔒 🔒 ? \*  
🔒 🔒 ? \*  
🔒 🔒 ? \*



Alice | 12345  
Bob | wordpass  
Eve | pa55word  
Joe | correcth..  
⋮  
Steve | hunter2

# Summary

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- 1 Secure computation:** Concepts & definitions
- 2 General constructions:** Yao's protocol, and GMW
- 3 Custom protocol:** private set intersection

Depending on the definition of “Function F”, MPC could be very powerful

# Materials

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- David Evans, Vladimir Kolesnikov and Mike Rosulek, [A Pragmatic Introduction to Secure Multi-Party Computation](#)
- Dan Boneh and Victor Shoup, [A Graduate Course in Applied Cryptography](#), Section 23

# Lecture 9: Privacy-Enhancing technologies 3: MPC

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**Diffie**



**Rivest**



**Rivest**



**Yao**



**Goldwasser**



**Hellman**



**Shamir**



**Adelman**



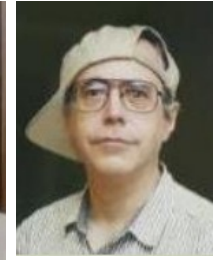
**Adelman**



**Dertouzos**



**Micali**



**Rackoff**

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**1976**

**New  
directions**

**1977**

**RSA**

**1978**

**Homomorphic Enc**

**1982**

**MPC**

**1985**

**Zero Knowledge**

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Thank you