Lecture 3: Public Key Cryptography

-COMP 6712 Advanced Security and Privacy

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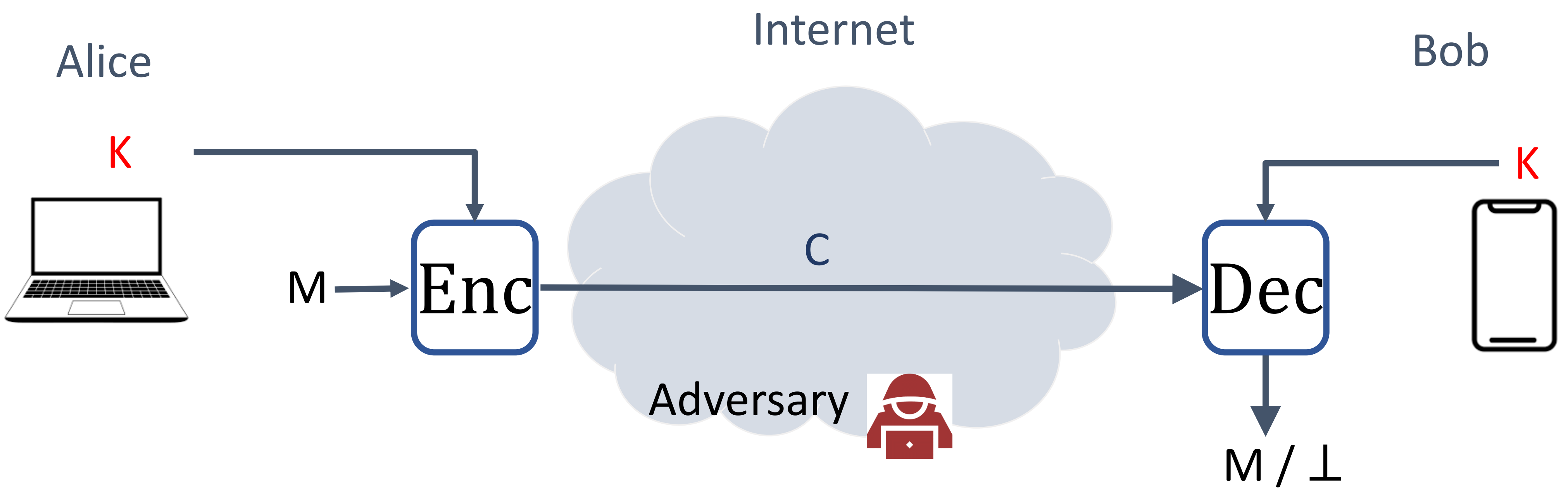
Public Key Cryptography

- Recall symmetric key cryptography (big picture)
- Diffie-Hellman Key Exchange
- Public key encryption: ElGamal, RSA
- Digital signature

Symmetric-key encryption



Enc : encryption algorithm (public)

K : shared key between Alice and Bob

Dec : decryption algorithm (public)

1.Kerckhoffs’ Principle (1883)

- Bob must have some information that Adversary doesn’t have

- How about keeping the decryption algorithm secret?
  - NO. algorithms for every user; share; need new design once broken

Design your system to be secure even if the attacker has complete knowledge of all its algorithms

- The only secret Bob has and Adversary doesn’t have is the SECRET KEY

2. Security definitions

- As said in lecture 2, we consider computational security (i.e., the adversary is computationally bounded)

Definition: A scheme Π is said to be **computationally secure** if any PPT adversary succeeds in breaking the scheme with negligible probability.

- But what is exactly mean by breaking?
- This is measured by the Aim and Capability of the adversary.

2. Security definitions

- Breaking/security is measured by the Aim and Capability of the adversary.

Aim

Try to learn something meaningful from the target ciphertext C^*

Capability

The ciphertext C^* + Learn more from system

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Aim

Try to learn something meaningful from the target ciphertext C^*

Given $C^* = \text{Enc}(m),\ f(m) \leftarrow A(C^*, .)$

$\Updownarrow$

A chooses any m_0, m_1
Given $C^* = \text{Enc}(m_b)$, Guess $b,\ b' \leftarrow A(C^*, .)$

Capability

The ciphertext C^* + Learn more from system

2. Security definitions

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Capability

The ciphertext C^* + Learn more from system

Only C^* XXX=eav

C^* and the adversary can choose plaintext
XXX=CPA; denoted by $A^{\mathrm{Enc}(\ )}$

C^* and adversary can further choose ciphertext
XXX=CCA; denoted by $A^{\mathrm{Enc}(\ ),\mathrm{Dec}(\ )}$

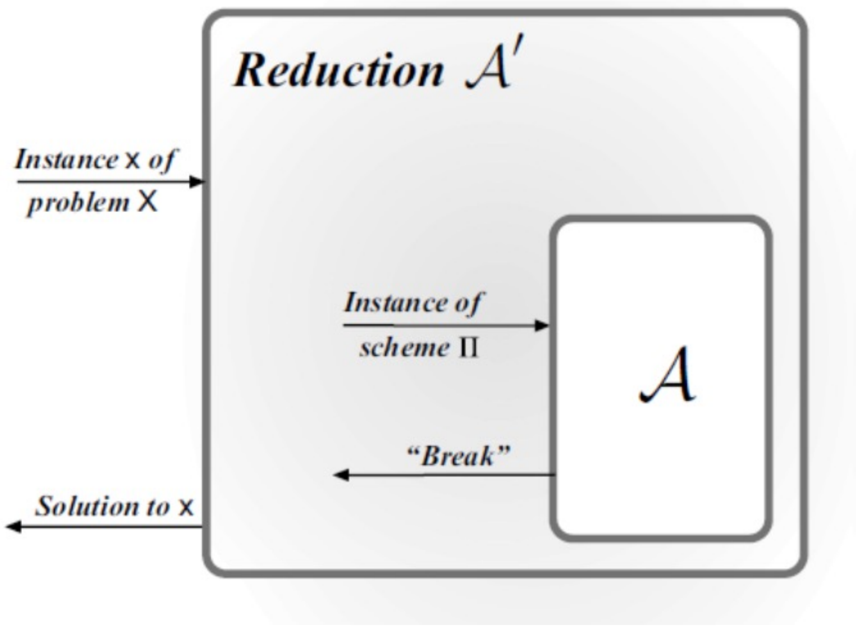
IND-XXX Security

3. Security Proof: reduction

Let us first talk about
how to show Problem A is harder than B?

Proving Π is secure is showing
Breaking Π is harder than Problem X

If Problem X is hard → Breaking Π is hard,
which means Π is secure

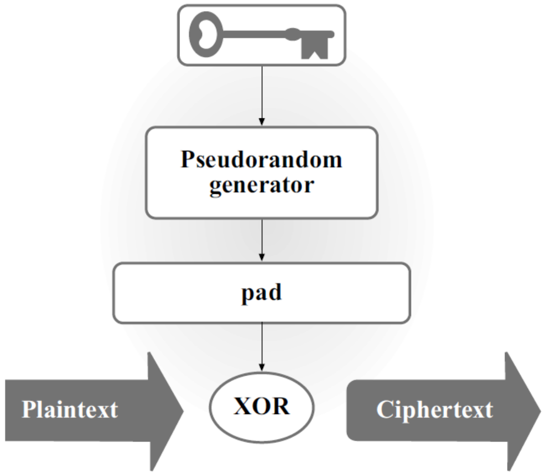


3. Security Proof: reduction: IND-eav as an example

$\Pi 1.\mathrm{Gen}: K \leftarrow \{0,1\}^k$

$\Pi 1.\mathrm{Enc}(\mathrm{K},\mathrm{M}): C = G(K) \oplus M$

$\Pi 1.\mathrm{Enc}(\mathrm{K},\mathrm{C}): M = G(K) \oplus C$



Test me on M_0, M_1

$C^* = G(K) \oplus M_b$

b'

3. Security Proof: reduction: IND-eav as an example

$\Pi 1.\mathrm{Gen}: K \leftarrow \{0,1\}^k$

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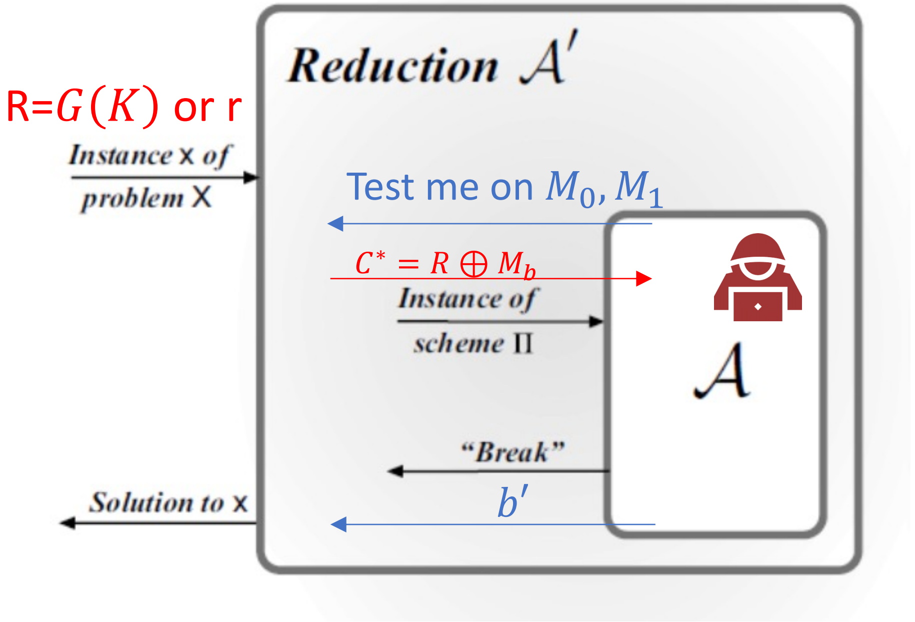
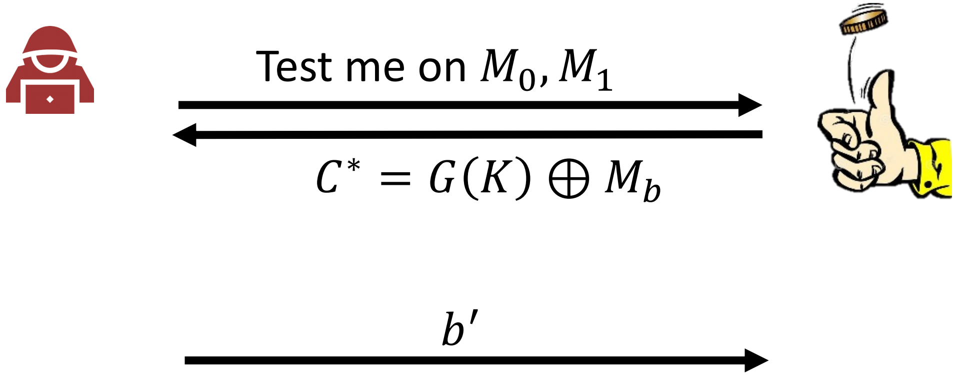
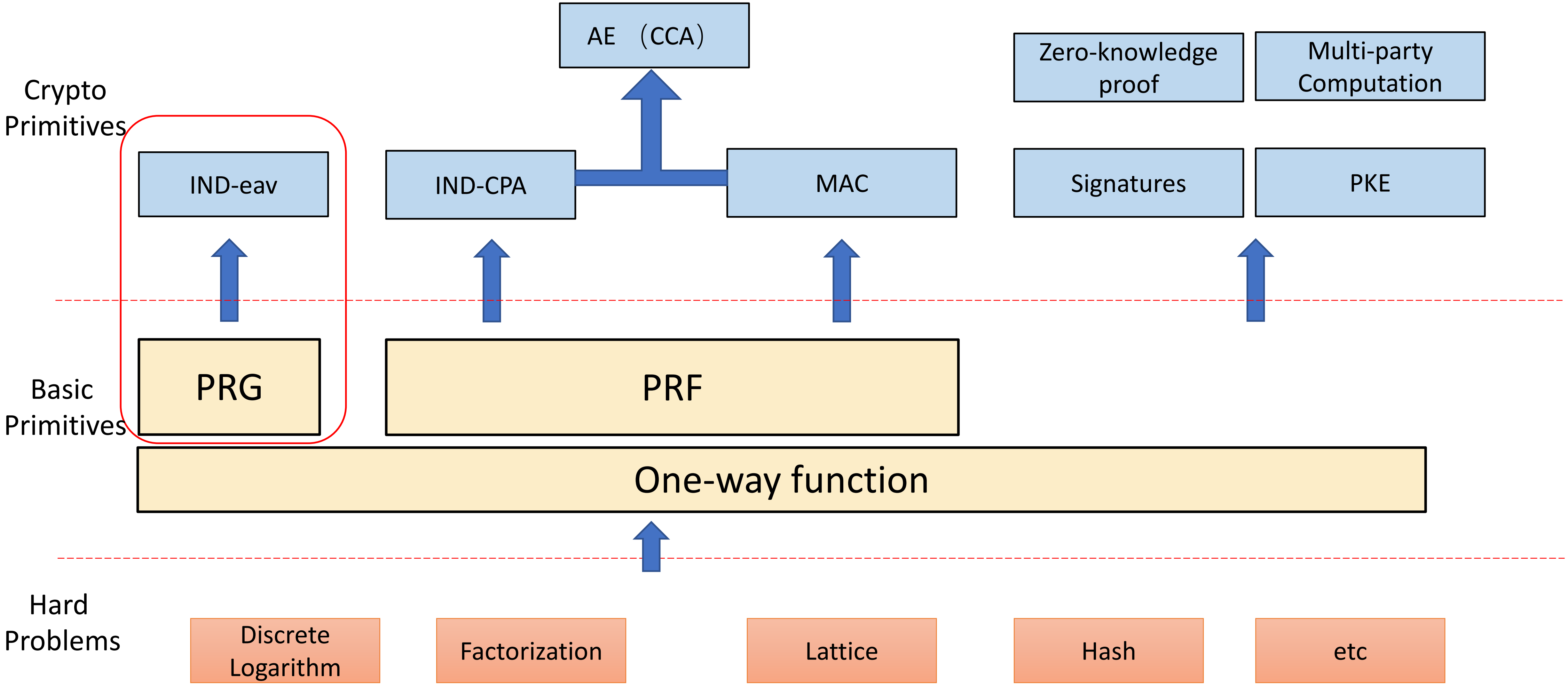
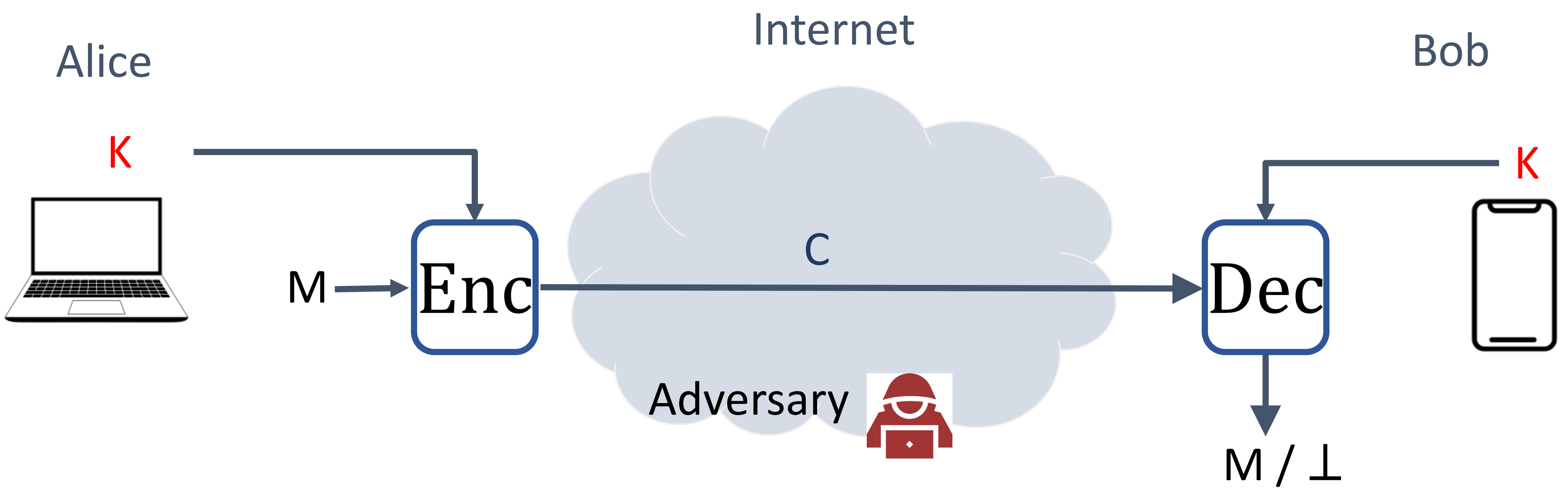




Big picture of Cryptography



Symmetric-key cryptography



Enc : encryption algorithm (public)

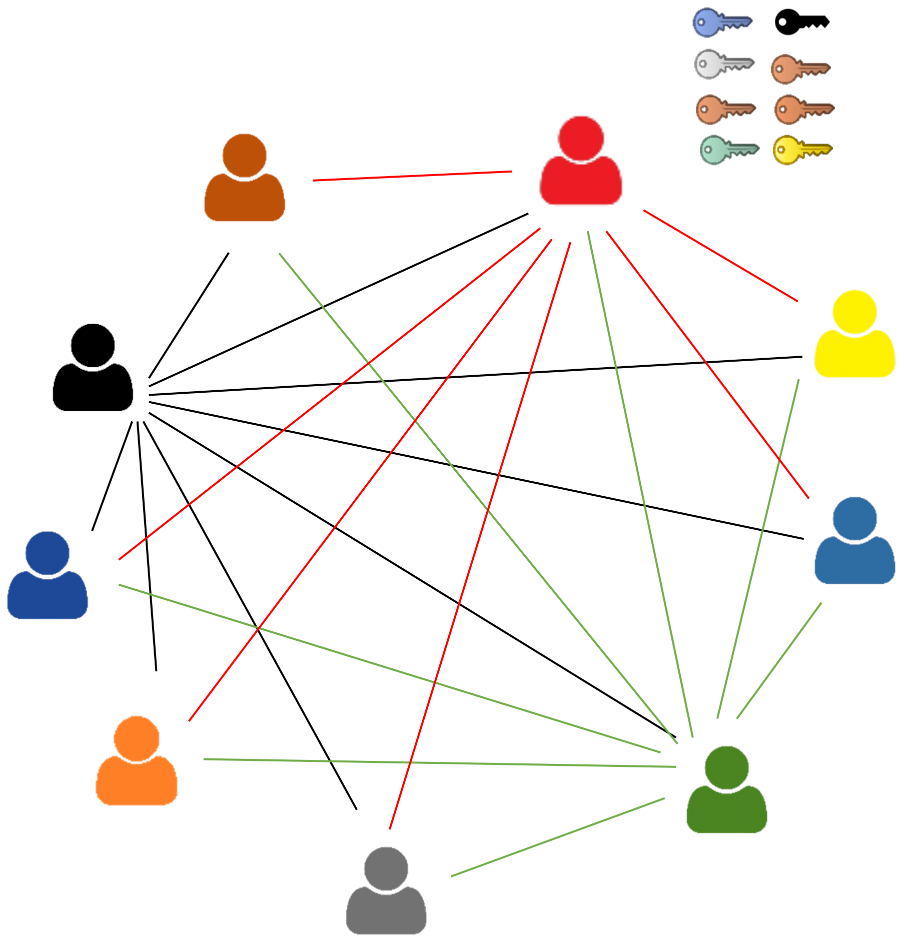
Dec : decryption algorithm (public)

K : shared key between Alice and Bob

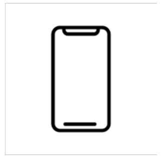
Ignore for now: How to achieve this??

Drawback of symmetric key

- One user needs to store N symmetric keys when communicating with N other users
- $\frac{N(N-1)}{2} = \mathcal{O}(N^2)$ keys in total
- Difficult to store and manage so many keys securely



$G = \langle g \rangle$



$A = g^{a} \longrightarrow$

$\longleftarrow B = g^{b}$

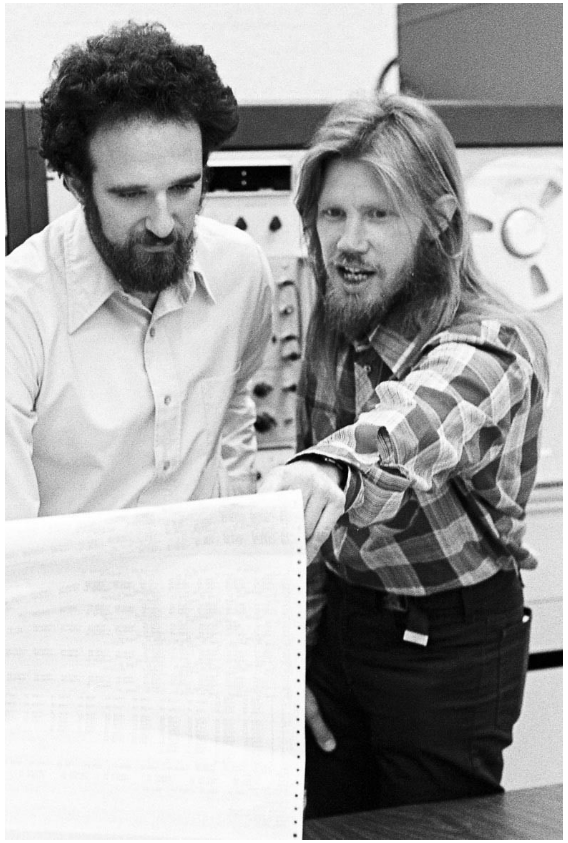
$K \leftarrow B^{a} = g^{ab}$

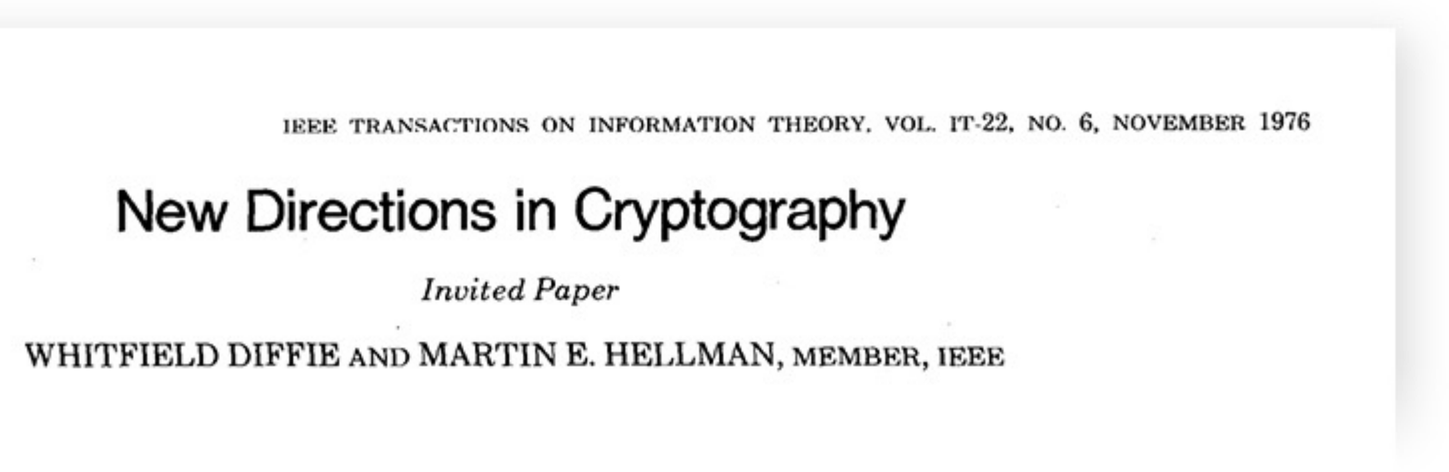
$K \leftarrow A^{b} = g^{ab}$

Diffie-Hellman Key Exchange

PKC

- Diffie-Hellman 1976 [New Directions in Cryptography]





IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. IT-22, NO. 6, NOVEMBER 1976

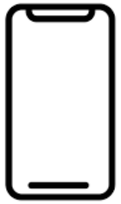
New Directions in Cryptography

Invited Paper

WHITFIELD DIFFIE AND MARTIN E. HELLMAN, MEMBER, IEEE

Diffie-Hellman Key Exchange

$G = \langle g \rangle$ ← public



$A = g^a$ →

$a \xleftarrow{\$} \{1, \dots, |G|\}$ ← $B = g^b$ $b \xleftarrow{\$} \{1, \dots, |G|\}$

$\mathrm{K} \leftarrow B^a = g^{ab}$

$\mathrm{K} \leftarrow A^b = g^{ab}$

Examples:

$G = (\mathbf{Z}_p^*, \cdot)$

$G = (E(\mathbf{Z}_p), +)$

G=($\boldsymbol{Z}_p^*$,⋅) preliminary

(integers) $\boldsymbol{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$

(integers "residue mod n") $\boldsymbol{Z}_n = \{0, 1, 2, \dots, n-1\}$

(integers "residue mod p") $\boldsymbol{Z}_p = \{0, 1, 2, \dots, p-1\}$ $\boldsymbol{Z}_p^* = \boldsymbol{Z}_p \setminus \{0\}$ p is a prime

Examples:

$$\boldsymbol{Z}_{11} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$\boldsymbol{Z}_{11}^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Define Group

Definition: A **group** $(G,\circ)$ is a set G together with a binary operation $\circ$ satisfying the following axioms.

$\boldsymbol{1}$: $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c, \in G$ (associativity)

$\boldsymbol{2}$: $\exists e \in G$ such that $e \circ a = a \circ e = a$ for all $a \in G$ (identity)

$\boldsymbol{3}$: $\forall a \in G$ there exists $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$ (inverse)

A group is **commutative** if: $a \circ b = b \circ a$ for all $a, b \in G$

The **order** of a group is the number of elements in G, denoted $|G|$

Examples

Definition: A **group** $(G,\circ)$ …

1: $(a \circ b) \circ c = a \circ (b \circ c)$ (associativity)
2: $\exists e \in G: e \circ a = a \circ e = a$ (identity)
3: $\exists a^{-1} \in G:\ a \circ a^{-1} = a^{-1} \circ a = e$ (inverse)

Groups

$(\boldsymbol{Z}, +)$ $e = 0$ $"3^{-1}" = -3$

$(\boldsymbol{Z}_n, +_n)$ $e = 0$ $"3^{-1}" = x\text{:}\ \ 3 + x \equiv 0 \bmod n$

$(\boldsymbol{Z}_p^*,\ \cdot_p)$ $e = 1$

$"3^{-1}" = x\text{:}\ \ 3 \cdot x \equiv 1 \bmod p$

When $\mathrm{p} = 5, "3^{-1}" = 2\text{:}\ \ 3 \cdot 2 \equiv 1 \bmod 5$

Not groups

$(\boldsymbol{Z}, \cdot)$ $2^{-1} = ?$

$(\boldsymbol{Z}, -)$ $(1 - 2) - 3 \neq 1 - (2 - 3)$

Group arithmetic

$g^0 \overset{\text{def}}{=} e$

$g^n \overset{\text{def}}{=} \overbrace{g \circ g \circ \cdots \circ g}^{n}$

$g^{-n} \overset{\text{def}}{=} (g^{-1})^n$

Fact: $g^n g^m = \underbrace{\overbrace{g \circ \cdots \circ g}^{n} \circ \overbrace{g \circ \cdots \circ g}^{m}}_{n+m} = g^{n+m}$

Fact: $(g^n)^m = g^{nm} = (g^m)^n$

$(\boldsymbol{Z}_7^*, \cdot)$

$3^5 \, mod \, 7 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 81 \cdot 3 \, mod \, 7 = 5$

Cyclic groups

Definition: A **group** $(G,\circ)$ is **cyclic** if there exists $g \in G$ such that

$$G = \{\, g^i \mid i \in \boldsymbol{Z} \,\} = \{\dots, g^{-2}, g^{-1}, g^0, g^1, g^2, g^3, \dots\}$$

Element g is called a **generator** for G and we write $(G,\circ) = \langle g \rangle$

Examples:

$$(\boldsymbol{Z}, +) = \langle 1 \rangle$$

$$(\boldsymbol{Z}_n, +_n) = \langle 1 \rangle$$

$$(\boldsymbol{Z}_p^*, \cdot) = \langle a \rangle$$

$$\begin{aligned}(\boldsymbol{Z}_7^*, \cdot) &= \langle 3 \rangle = \{3^0, 3^1, 3^2, 3^3, 3^4, 3^5\} = \{1, 3, 2, 6, 4, 5\} \\ &= \langle 5 \rangle = \{5^0, 5^1, 5^2, 5^3, 5^4, 5^5\} = \{1, 5, 4, 6, 2, 3\} \\ &\neq \langle 2 \rangle = \{2^0, 2^1, 2^2, 2^3, 2^4, 2^5\} = \{1, 2, 4, 1, 2, 4\} = \{1, 2, 4\}\end{aligned}$$

Cyclic groups

Theorem: if $(G,\circ)$ is a finite group, then for all $g \in G$:

$$g^{|G|} = e$$

Proof (finite cyclic groups):

$$|G| = |\langle g \rangle| = n$$

$$\overbrace{e \quad g^1 \quad g^2 \quad g^3 \quad \cdots \quad g^{n-1}}^{G} \quad g^n \quad g^{n+1} \quad g^{n+2} \quad \cdots$$

$$g^n = g^3 \quad \Longrightarrow \quad g^{n-3} = e \quad \Longrightarrow \quad g^j = e \quad j < n$$ contradiction!

Corollary I: $g^i = g^{i \bmod n} = g^{i \bmod |G|}$

$$(\boldsymbol{Z}_7^*, \cdot) = \langle 3 \rangle = \{3^0, 3^1, 3^2, 3^3, 3^4, 3^5\} = \{1, 3, 2, 6, 4, 5\}$$
$$= \langle 5 \rangle = \{5^0, 5^1, 5^2, 5^3, 5^4, 5^5\} = \{1, 5, 4, 6, 2, 3\}$$
$$\neq \langle 2 \rangle = \{2^0, 2^1, 2^2, 2^3, 2^4, 2^5\} = \{1, 2, 4, 1, 2, 4\} = \{1, 2, 4\}$$

$\langle 2 \rangle$ is a sub-group of $(\boldsymbol{Z}_7^*, \cdot)$ with order 3

Suppose $p = 2q + 1$, with q being prime. $(\boldsymbol{Z}_p^*, \cdot)$ has a sub-group $\langle g \rangle$ of order q

Denoted by $\langle g \rangle < \boldsymbol{Z}_p^*$

Example: $\boldsymbol{Z}_{11}^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

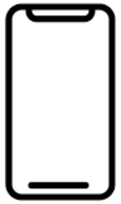
$11 = 2 \cdot 5 + 1$

For $g = 3,4,5,9$, $\langle 3 \rangle = \langle 4 \rangle = \langle 5 \rangle = \langle 9 \rangle = \{1, 3, 4, 5, 9\} < \boldsymbol{Z}_{11}^*$

https://sagecell.sagemath.org/

Diffie-Hellman Key Exchange

$G = \langle g \rangle$ ← public



$A = g^{a}$ →

← $B = g^{b}$

$a \xleftarrow{\$} \{1, \dots, |G|\}$

$b \xleftarrow{\$} \{1, \dots, |G|\}$

$\mathrm{K} \leftarrow B^{a} = g^{ab}$

$\mathrm{K} \leftarrow A^{b} = g^{ab}$

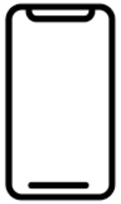
Examples:

$G = (\mathbf{Z}_p^*, \cdot)$

Diffie-Hellman Key Exchange

public

$$G = \langle g = 3 \rangle$$



$$\color{red}{A = 9} = 3^{\color{red}{2}} \text{ mod } 11 \longrightarrow$$

$$\longleftarrow \color{green}{B = 5} = 3^{\color{green}{3}} \text{ mod } 11$$

$$\color{red}{2} \overset{\$}{\leftarrow} \{1, \ldots, 5\} \qquad \color{green}{3} \overset{\$}{\leftarrow} \{1, \ldots, 5\}$$

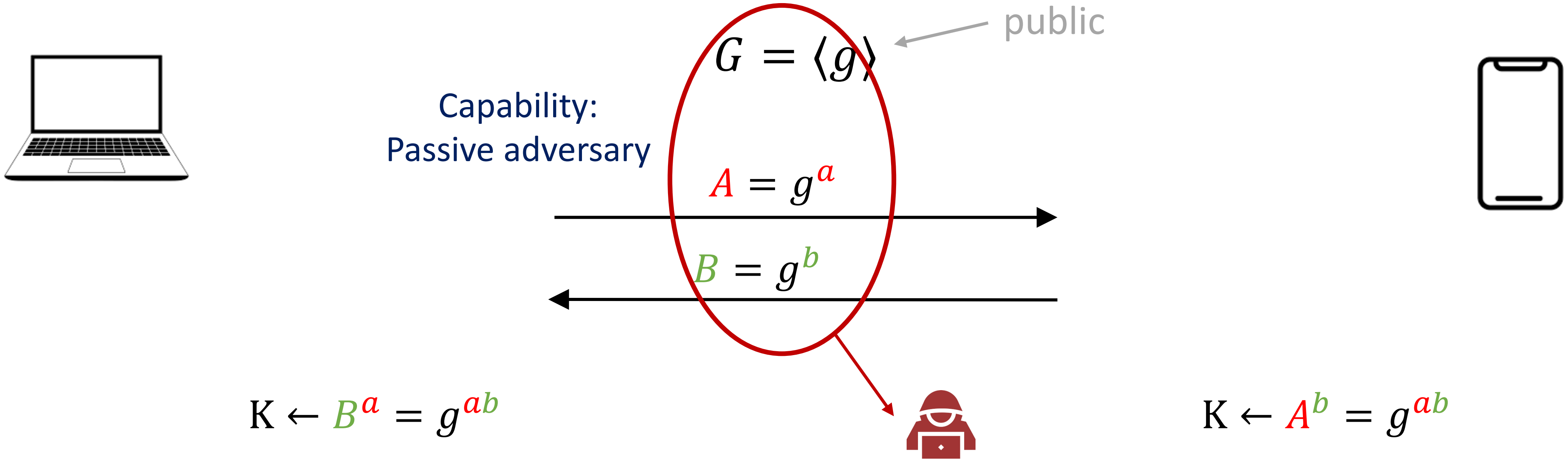
$$\mathrm{K} = 4 \leftarrow \color{green}{B}^{\color{red}{a}} = g^{\color{red}{a}\color{green}{b}} \qquad \mathrm{K} = 4 \leftarrow \color{red}{A}^{\color{green}{b}} = g^{\color{red}{a}\color{green}{b}}$$

Exp. $G = (\boldsymbol{Z}_p^*, \cdot), p = 11, g = 3$

To be secure: length p must be large

https://www.rfc-editor.org/rfc/rfc2409#section-6.2; rfc3526#page-3

Diffie-Hellman Key Exchange



Aim: $K \approx_c$ random key???

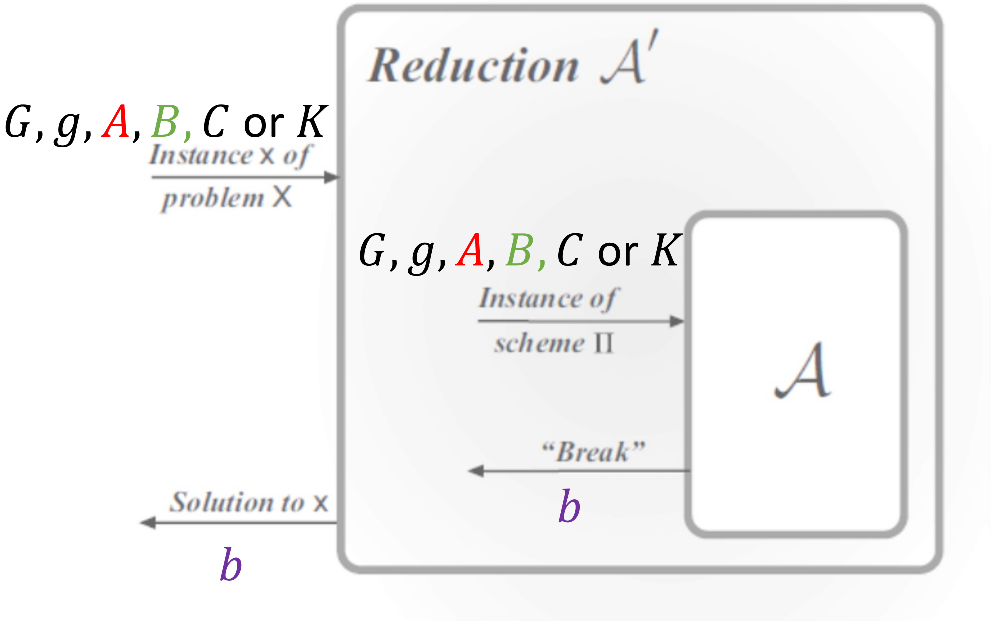
Security (given G, g, A, B):

- Must be hard to distinguish $K \leftarrow g^{ab}$ from random key

Security under DH assumption

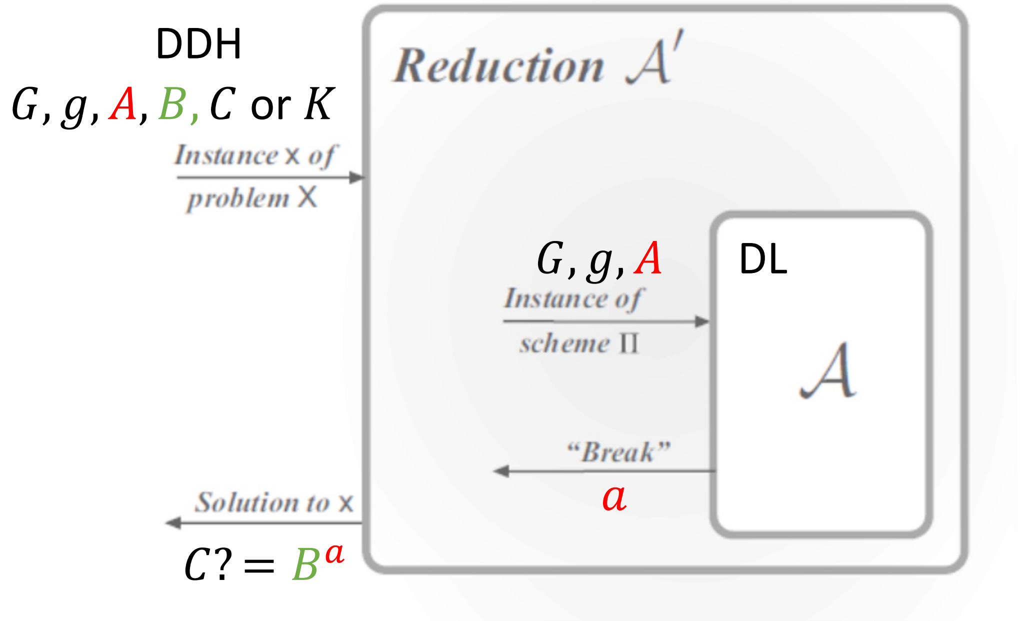
DDH assumption: **given** G, g, A, B**:**

- Must be hard to distinguish $\mathrm{K} \leftarrow g^{ab}$ from random key C



Discrete logarithm (DL) assumption

Discrete logarithm assumption: **given** G, g, A**:**
- it is hard to find a such that $A = g^a$



DDH assumption: **given** G, g, A, B**:**
- Must be hard to distinguish $\mathrm{K} \leftarrow g^{ab}$ from random key C

DL ≥ DDH

To let DDH assumption holds, $|< g >|$ should be large

Diffie-Hellman –$(\boldsymbol{Z}_p^*,\cdot)$ -group 14 of RFC 3526

$$p = \begin{array}{l}3231700607131100730033891392642382824881794124114023911284200975140074170663435422261968941736356934711790173790970419175460587320919502885375898618562215321217541251490177452027023579607823624888424618947758764\,1\\1059286460994117232454266225221932305409190376805242355191256797158701170010580558776510388618472802579760549035697325615261670813393617995413364765591603683178967290731783845896806396719009772021941686472258710\,3\\1411336429319536193471636533209717077448227988588565369208645296636077250268955505928362751121174096972998068410554359584866583291642136218231078990999448652468262416972035911852507045361090559\end{array}$$

$$= 2 \cdot q + 1$$

$$\langle g \rangle = \langle 2 \rangle < (\boldsymbol{Z}_p^*,\cdot)$$

$$\begin{array}{l}4133497276497869747688813147795369159059830288804878914\\1980374225868022299895295637179960494385138311397470827\\3495908499817923294866040090426152978689798973304849\\2192783122517759578426588905583969680803171945290768251\\3329106414714573595610370323158171874543264533536374242\\4060090852307081678111805222242680612035529830764495\\4854397711957145664127387682101302285070146685335073463\\4588800753314418345842915139086907784354458905455294406\\1903282023858937253321947252641343640598633640822291\\6074934143291019888804465974702355801616825216730698060\\5647973065295774322785381793841598013402958235279155806\\13351187034853959149\end{array} \overset{\$}{\leftarrow} \{1 \ldots q\}$$

$$A = 2^{\begin{array}{l}4133497276497869747688813147795369159059830288804878914\\1980374225868022299895295637179960494385138311397470827\\3495908499817923294866040090426152978689798973304849\\2192783122517759578426588905583969680803171945290768251\\3329106414714573595610370323158171874543264533536374242\\4060090852307081678111805222242680612035529830764495\\4854397711957145664127387682101302285070146685335073463\\4588800753314418345842915139086907784354458905455294406\\1903282023858937253321947252641343640598633640822291\\6074934143291019888804465974702355801616825216730698060\\5647973065295774322785381793841598013402958235279155806\\13351187034853959149\end{array}} \bmod p \longrightarrow$$

$$\longleftarrow B = 2^{\begin{array}{l}4723789759653965825188322563356453187611707360965359685\\7880669992230757150404954632247448495838204004394852892\\9671269507109245266174621284477799937918964628789631\\2348152272627069323792055856790321199248897271347298277\\2848744522640803022909928099136539284836286481767224130\\5932059800017497039892171592547336108905906405436246\\6987620661784155427177071979138656350318731235462967486\\0703821404739110104286042063247209755506195200644989056\\1683478362740082015762982050288677324025573804780149\\8030979920739061611583799751934000077568119763119040673\\1683727944709941956370245115081620783256133515159656005\\7242643342201291440\end{array}} \bmod p$$

$$\begin{array}{l}4723789759653965825188322563356453187611707360965359685\\7880669992230757150404954632247448495838204004394852892\\9671269507109245266174621284477799937918964628789631\\2348152272627069323792055856790321199248897271347298277\\2848744522640803022909928099136539284836286481767224130\\5932059800017497039892171592547336108905906405436246\\6987620661784155427177071979138656350318731235462967486\\0703821404739110104286042063247209755506195200644989056\\1683478362740082015762982050288677324025573804780149\\8030979920739061611583799751934000077568119763119040673\\1683727944709941956370245115081620783256133515159656005\\7242643342201291440\end{array} \overset{\$}{\leftarrow} \{1 \ldots q\}$$

Corollary I: $g^i = g^{i \bmod |H|}$

$$Z \leftarrow 2^{\begin{array}{l}4133497276497869747688813147795369159059830288804878914\\1980374225868022299895295637179960494385138311397470827\\3495908499817923294866040090426152978689798973304849\\2192783122517759578426588905583969680803171945290768251\\3329106414714573595610370323158171874543264533536374242\\4060090852307081678111805222242680612035529830764495\\4854397711957145664127387682101302285070146685335073463\\4588800753314418345842915139086907784354458905455294406\\1903282023858937253321947252641343640598633640822291\\6074934143291019888804465974702355801616825216730698060\\5647973065295774322785381793841598013402958235279155806\\13351187034853959149\end{array} \times \begin{array}{l}4723789759653965825188322563356453187611707360965359685\\7880669992230757150404954632247448495838204004394852892\\9671269507109245266174621284477799937918964628789631\\2348152272627069323792055856790321199248897271347298277\\2848744522640803022909928099136539284836286481767224130\\5932059800017497039892171592547336108905906405436246\\6987620661784155427177071979138656350318731235462967486\\0703821404739110104286042063247209755506195200644989056\\1683478362740082015762982050288677324025573804780149\\8030979920739061611583799751934000077568119763119040673\\1683727944709941956370245115081620783256133515159656005\\7242643342201291440\end{array}} \bmod p$$

Demo

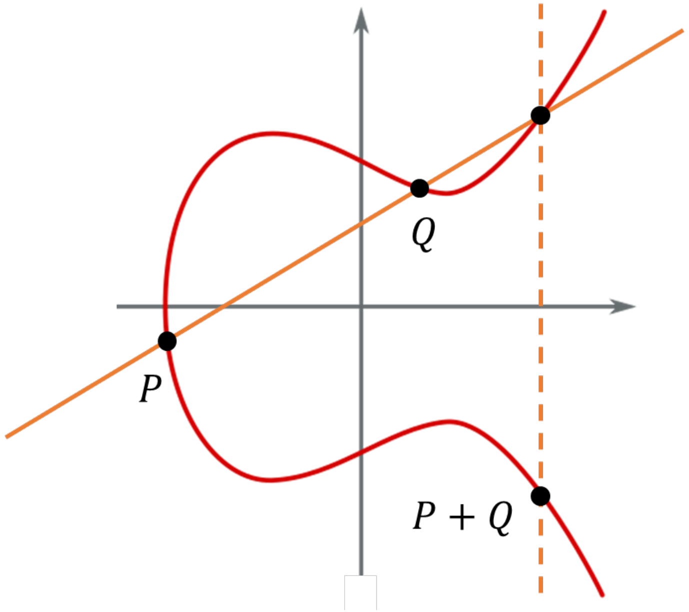
- RFC 3526
- Demonstration using SageMath
- https://sagecell.sagemath.org/

Better alternatives to $\boldsymbol{Z}_p^*$?

Elliptic curves

$$y^2 = x^3 + ax + b$$

$$a, b, x, y \in \boldsymbol{R}$$



- There is elliptic curves defined over $\boldsymbol{Z}_p$
- Such that the points on an elliptic curve (+ a infinite point) form a group of order $\sim p^2$
- Denoted by $(E(\boldsymbol{Z}_p), +)$

Cryptographic groups in practice

- $(\boldsymbol{Z}_p^*, \cdot)$ groups:
  - **TLS 1.3:** five specific groups allowed
    - size $\approx 2^{2048}, 2^{3072}, 2^{4096}, 2^{6144}, 2^{8192}$ (RFC 7919)
  - **IKEv2** (IPsec key exchange protocol): MODP groups
    - size $\approx 2^{768}, 2^{1024}, 2^{1536}, 2^{2048}, 2^{3072}, 2^{4096}, 2^{6144}, 2^{8192}$ (RFC 7296 and RFC 3526)
  - all p's are **safe primes** (i.e., of the form $p = 2q + 1$ where q is prime)

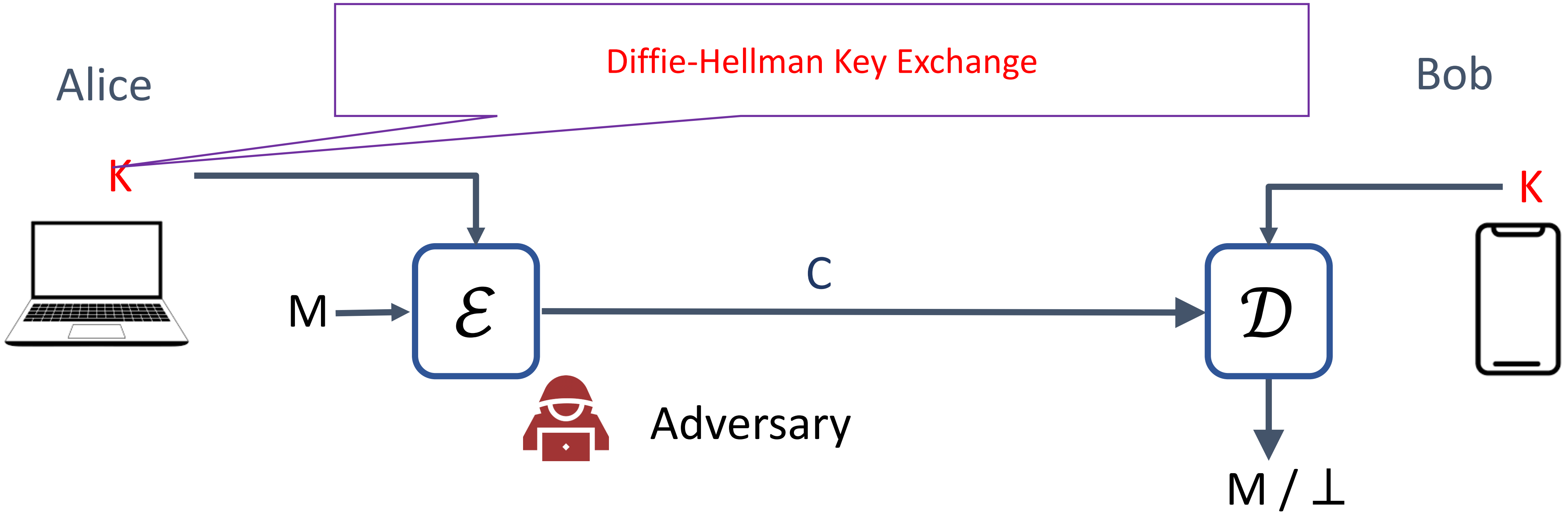
- $(E(\boldsymbol{Z}_p), +)$ groups
  - NIST groups: P-224, P-256, P-384, P-521
  - Curve25519 ($E : y^2 = x^3 + 486662x^2 + x$ and $p = 2^{255} - 19$) (Daniel J. Bernstein)
  - Curve448 ($E : y^2 + x^2 = 1 - 39081x^2y^2$ and $p = 2^{448} - 2^{224} - 1$) (Mike Hamburg)

A short summary

- Diffie-Hellman Key Exchange could help to share a secret
- Using group $(\boldsymbol{Z}_p^*, \cdot)$ or $(E(\boldsymbol{Z}_p), +)$
- DH problem is the underlying hard problem

Public key encryption

Diffie-Hellman then Symmetric-key cryptography

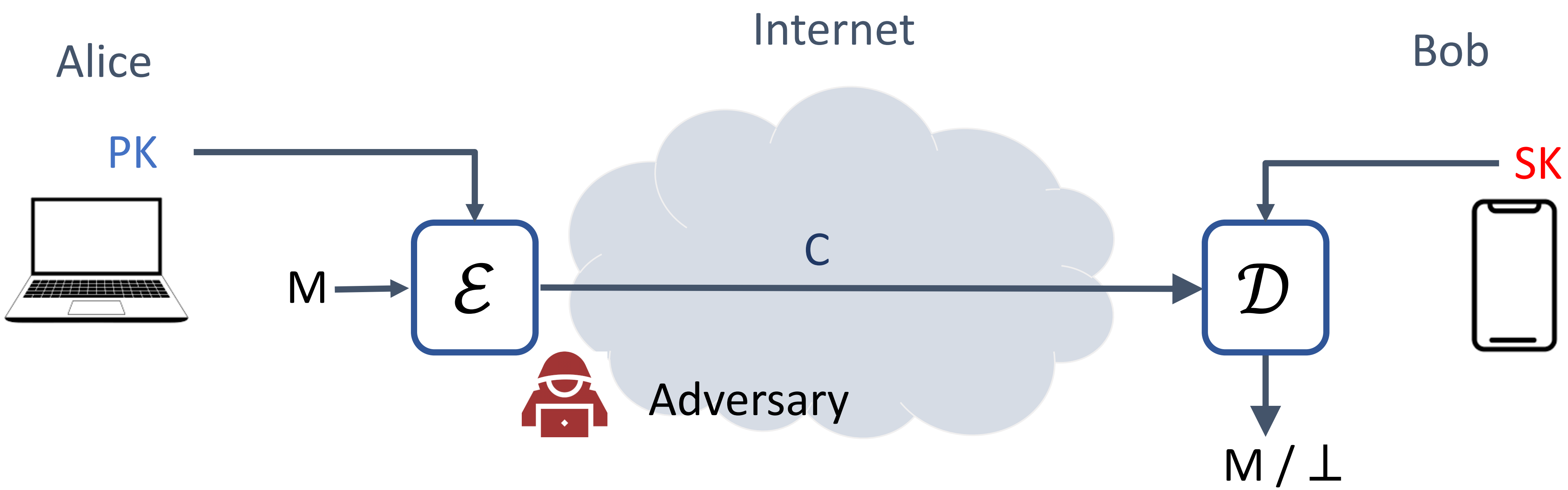


Enc: encryption algorithm (public)

K : shared key between Alice and Bob

Dec: decryption algorithm (public)

Public-key Encryption directly???



Enc: encryption algorithm (public)

PK : public key of Bob (public)

Dec : decryption algorithm (public)

SK : secret key (secret)

Public-key encryption – syntax

A **public-key encryption scheme** is a tuple $\Sigma = (\mathrm{KeyGen}, \mathrm{Enc}, \mathrm{Dec})$ of algorithms

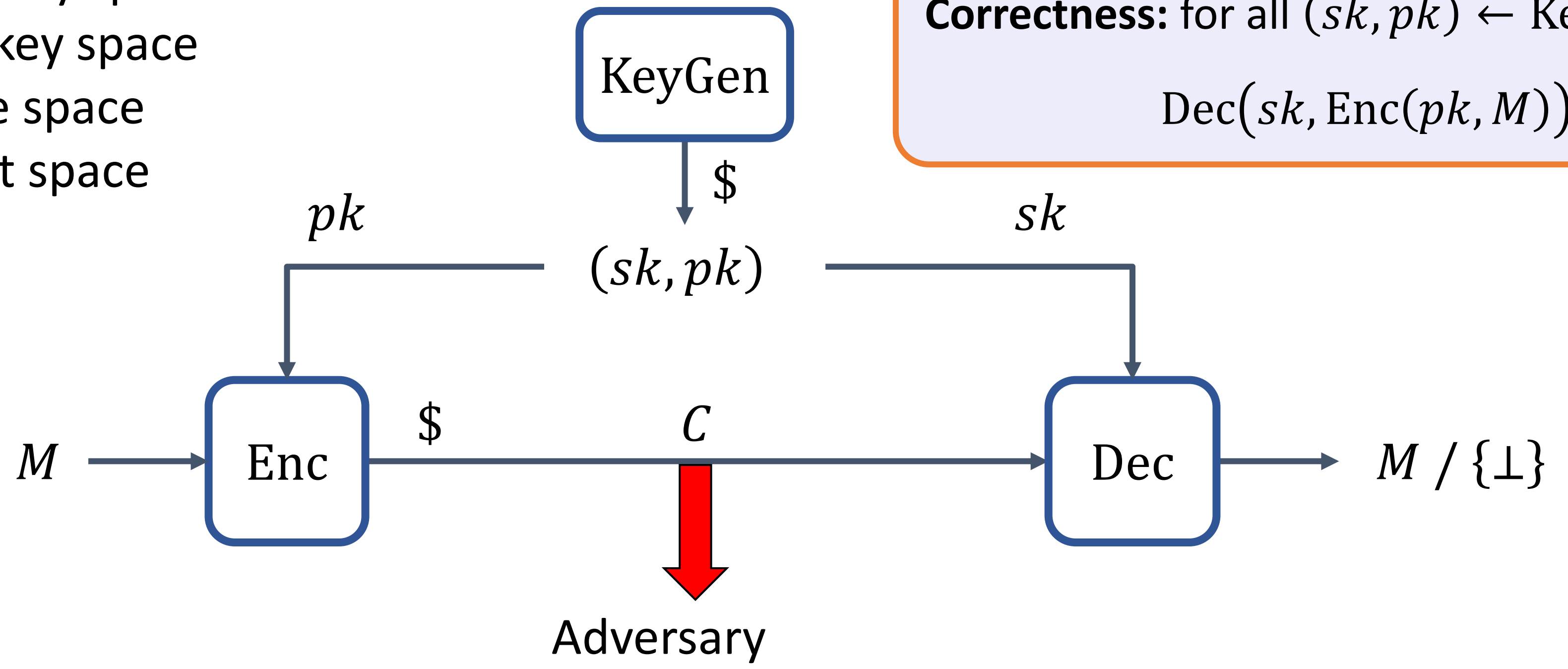
$$(sk, pk) \xleftarrow{\$} \mathrm{KeyGen} \qquad \mathrm{Enc} : \mathcal{PK} \times \mathcal{M} \rightarrow \mathcal{C} \qquad \mathrm{Dec} : \mathcal{SK} \times \mathcal{C} \rightarrow \mathcal{M} \cup \{\perp\}$$

$$\mathrm{Enc}(pk, M) = \mathrm{Enc}_{pk}(M) = C \qquad \mathrm{Dec}(sk, C) = \mathrm{Dec}_{SK}(C) = M/\perp$$

- $\mathcal{SK}$ – private key space
- $\mathcal{PK}$ – public key space
- $\mathcal{M}$ – message space
- $\mathcal{C}$ – ciphertext space

Correctness: for all $(sk, pk) \leftarrow \mathrm{KeyGen}$:

$$\mathrm{Dec}\big(sk, \mathrm{Enc}(pk, M)\big) = M$$



Public-key encryption – security: IND-CPA

$\mathbf{Exp}_{\Sigma}^{\text{ind-cpa}}(A)$

$b \xleftarrow{\$} \{0,1\}$

$(sk, pk) \xleftarrow{\$} \Sigma.\mathrm{KeyGen}$

$M_0^*, M_1^* \leftarrow A(pk)$ // find stage

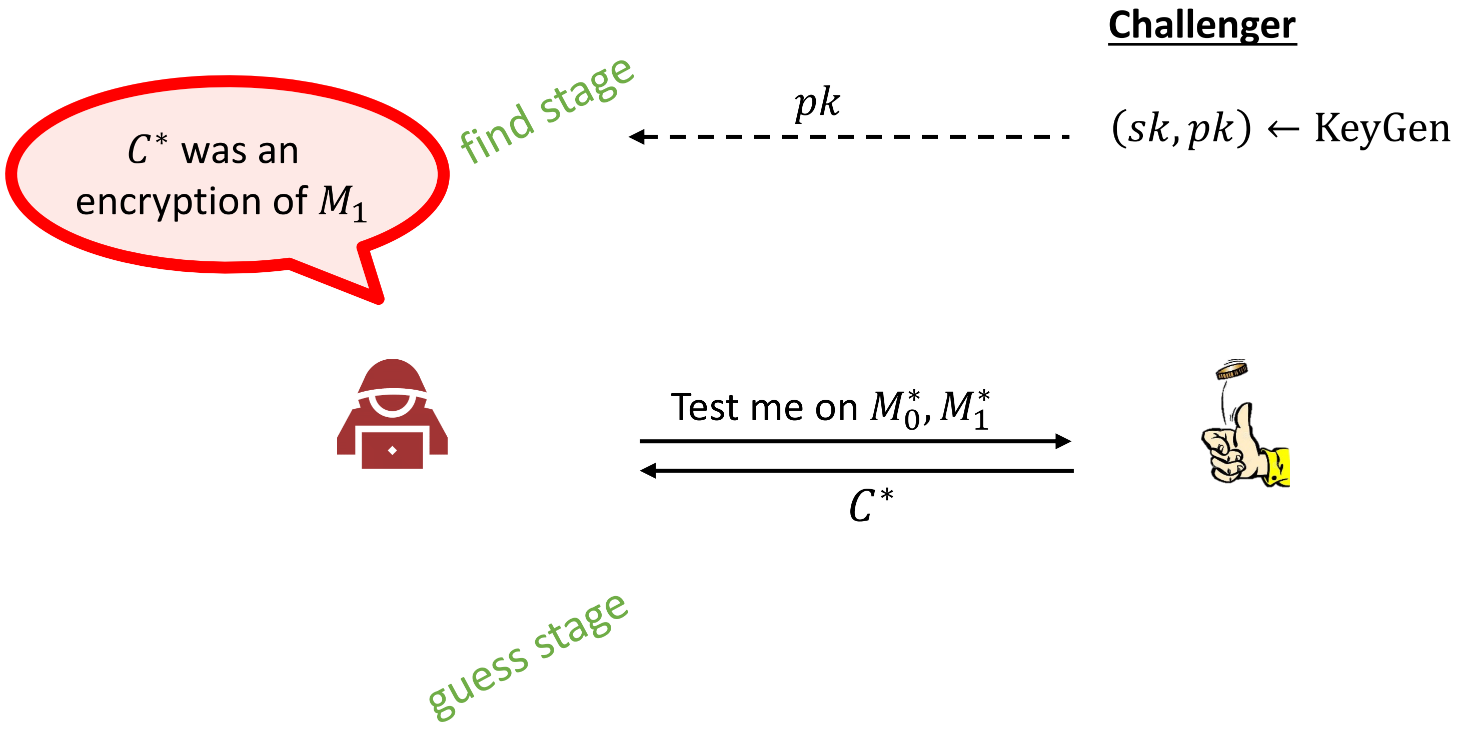
if $|M_0^*| \neq |M_1^*|$ **then**

  return $\perp$

$C^* \leftarrow \Sigma.\mathrm{Enc}(pk, M_b^*)$

$b' \leftarrow A(pk, C^*)$ // guess stage

return $b' \stackrel{?}{=} b$



Definition: The **IND-CPA-advantage** of an adversary A is

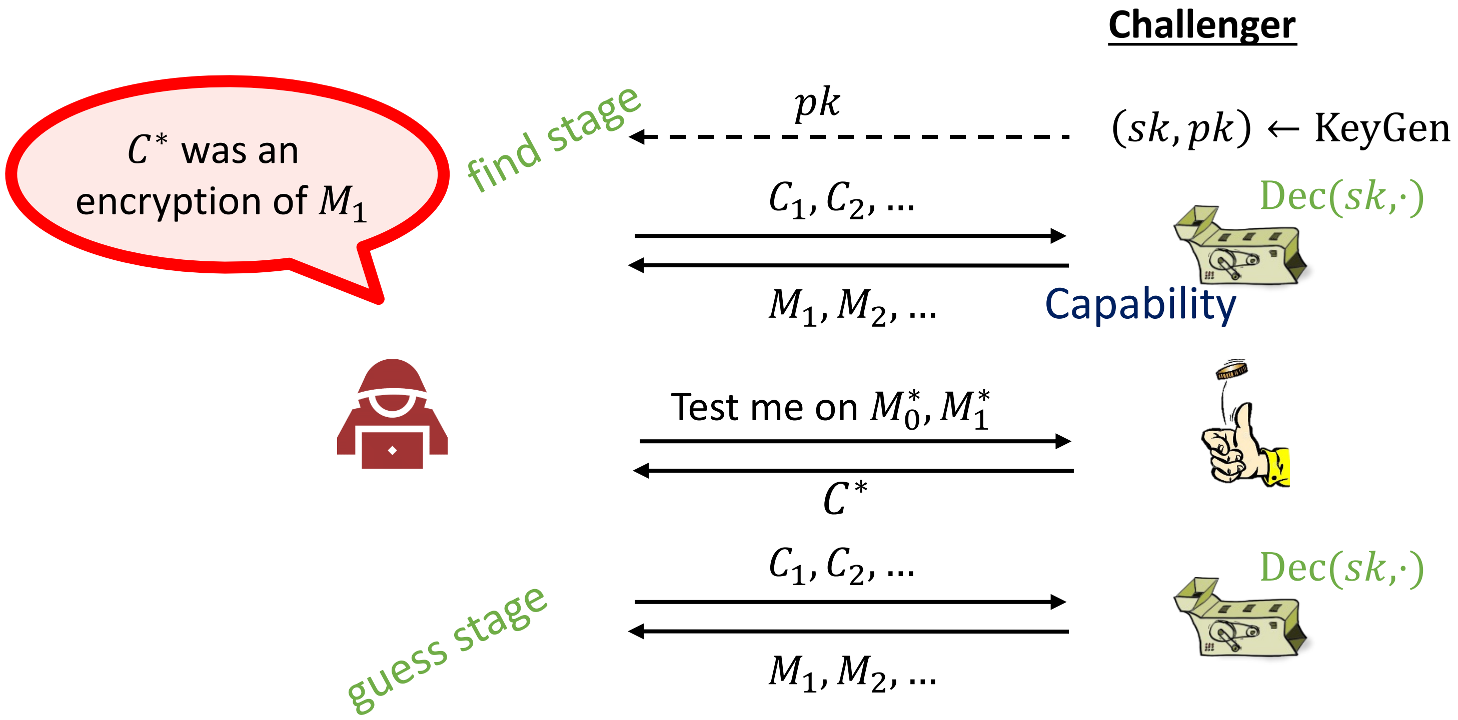
$$\mathbf{Adv}_{\Sigma}^{\text{ind-cpa}}(A) = \left|2 \cdot \Pr\left[\mathbf{Exp}_{\Sigma}^{\text{ind-cpa}}(A) \Rightarrow \mathrm{true}\right] - 1\right|$$

Public-key encryption – security: IND-CCA

$\mathbf{Exp}_{\Sigma}^{\text{ind-cca}}(A)$

$b \xleftarrow{\$} \{0,1\}$
$(sk, pk) \xleftarrow{\$} \Sigma.\mathrm{KeyGen}$
$M_0^*, M_1^* \leftarrow A^{\mathcal{D}_{sk}(\cdot)}(pk)$ // find stage
if $|M_0^*| \neq |M_1^*|$ **then**
  return $\perp$
$C^* \leftarrow \Sigma.\mathrm{Enc}(pk, M_b^*)$
$b' \leftarrow A^{\mathcal{D}_{sk}(\cdot)}(pk, C^*)$ // guess stage
return $b' \overset{?}{=} b$

$\mathcal{D}_{sk}(C)$
----
if $C = C^*$ **the** // cheating!
  return $\perp$
return $\Sigma.\mathrm{Dec}(sk, C)$



Definition: The **IND-CCA-advantage** of an adversary A is

$$\mathbf{Adv}_{\Sigma}^{\text{ind-cca}}(A) = \left|2 \cdot \Pr\left[\mathbf{Exp}_{\Sigma}^{\text{ind-cca}}(A) \Rightarrow \text{true}\right] - 1\right|$$

Scheme ElGamal

$$G = \langle g \rangle$$

$$\xrightarrow{\quad A \quad}$$

$$\xleftarrow{\quad B \quad}$$

$b \xleftarrow{\$} \{1, \dots, |G|\}$

$B \leftarrow g^{b}$

$a \xleftarrow{\$} \{1, \dots, |G|\}$

$A \leftarrow g^{a}$

$K \leftarrow B^{a} = g^{ab}$

$K \leftarrow A^{b} = g^{ab}$

ElGamal

$$G = \langle g \rangle$$

$$b \xleftarrow{\$} \{1, \dots, |G|\}$$

$$B \leftarrow g^b$$

$$\xleftarrow{\quad B \quad}$$

$$\xrightarrow{\quad A \quad}$$

$$a \xleftarrow{\$} \{1, \dots, |G|\}$$

$$A \leftarrow g^a$$

$$K \leftarrow B^a = g^{ab}$$

$$K \leftarrow A^b = g^{ab}$$

ElGamal

$$G = \langle g \rangle$$

$b \xleftarrow{\$} \{1, \dots, |G|\}$

$B \leftarrow g^b$

$$\xleftarrow{\quad B \quad}$$

$$\xrightarrow{\quad A, C \quad}$$

$a \xleftarrow{\$} \{1, \dots, |G|\}$

$A \leftarrow g^a$

$K \leftarrow B^a = g^{ab}$

$C \leftarrow K \cdot M$

$K \leftarrow A^b = g^{ab}$

$M \leftarrow C/K$

ElGamal

$$G = \langle g \rangle$$

KeyGen

1. $sk = b \xleftarrow{\$} \{1, \dots, |G|\}$
2. $pk = B \leftarrow g^b$
3. **return** (sk, pk)

B

A, C

$a \xleftarrow{\$} \{1, \dots, |G|\}$

$A \leftarrow g^a$

$K \leftarrow B^a = g^{ab}$

$C \leftarrow K \cdot M$

$K \leftarrow A^b = g^{ab}$

$M \leftarrow C/K$

ElGamal

$$G = \langle g \rangle$$

$$\xleftarrow{\quad B \quad}$$

$$\xrightarrow{\quad A, C \quad}$$

KeyGen

1. $sk = b \xleftarrow{\$} \{1, \dots, |G|\}$
2. $pk = B \leftarrow g^b$
3. **return** (sk, pk)

Enc(pk, M)

1. $a \xleftarrow{\$} \{1, \dots, |G|\}$
2. $A \leftarrow g^a$
3. $K \leftarrow B^a = g^{ab}$
4. $C \leftarrow K \cdot M$
5. **return** (A, C)

$Z \leftarrow A^b = g^{ab}$

$M \leftarrow C/K$

ElGamal

$\text{ElGamal.Enc}: G \times G \to \quad G \times \mathcal{C}$

$\text{ElGamal.Dec}: \boldsymbol{Z}_p \times G \times G \to \quad G$

$G = \langle g \rangle$

B

A, C

KeyGen

1. $sk = \quad b \xleftarrow{\$} \{1, \ldots, |G|\}$
2. $pk = \quad B \leftarrow g^b$
3. **return** (sk, pk)

Enc(pk, M)

1. $a \xleftarrow{\$} \{1, \ldots, |G|\}$
2. $A \leftarrow g^a$
3. $K \leftarrow B^a = g^{ab}$
4. $C \leftarrow K \cdot M$
5. **return** (A, C)

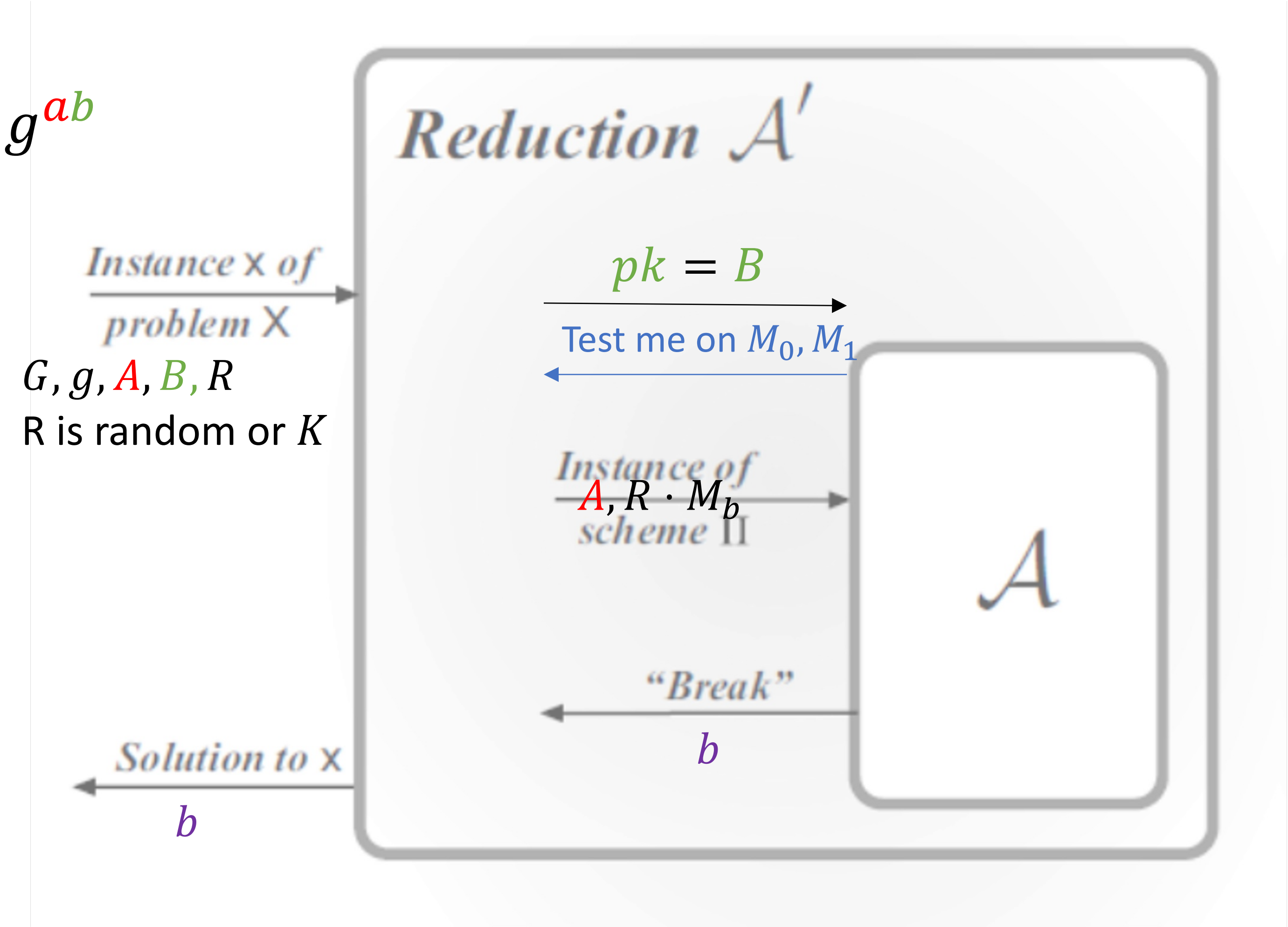
Dec(sk, C)

1. $Z \leftarrow A^b = g^{ab}$
2. $M \leftarrow C/K$
3. **return** M

ElGamal is IND-CPA under DDH assumption

DDH assumption: **given** G, g, A, B**:**

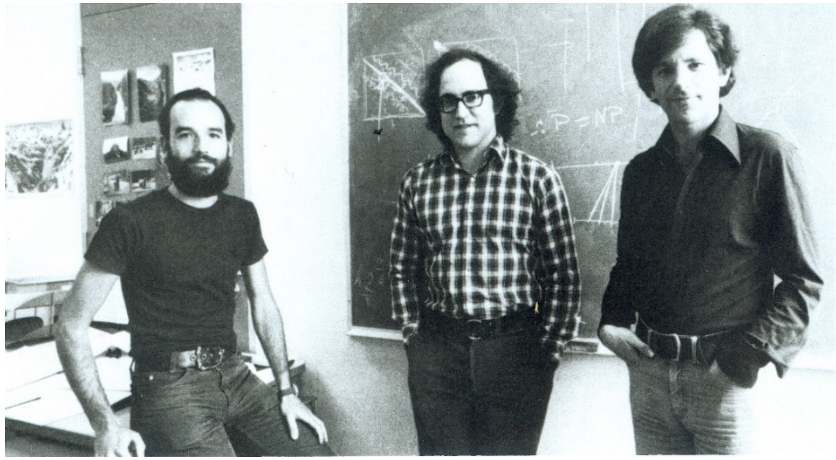
- Must be hard to distinguish $\mathrm{K} \leftarrow g^{ab}$ from random key R



RSA in 1977

- The RSA encryption scheme

$$c = E\,(m) = m^e \pmod N$$



Adi Shamir Ron Rivest Leonard Adleman

The group $(\boldsymbol{Z}_n^*,\cdot)$

$\boldsymbol{Z}_p = \{0, 1, \dots, p-1\}$ $\quad$ $(\boldsymbol{Z}_p,\cdot)$ is *not* a group!

$\boldsymbol{Z}_p^* = \{1, \dots, p-1\}$ $\quad$ $(\boldsymbol{Z}_p^*,\cdot)$ *is* a group!

$\boldsymbol{Z}_n = \{0, 1, \dots, n-1\}$ $\quad$ $(\boldsymbol{Z}_n,\cdot)$ is *not* a group!

$\boldsymbol{Z}_n^* \neq \underbrace{\{1, \dots, n-1\}}_{\boldsymbol{Z}_n^+}$ $\quad$ $(\boldsymbol{Z}_n^+,\cdot)$ is *also not* a group!

$$\boldsymbol{Z}_n^* = \underbrace{\text{invertible elements in } \boldsymbol{Z}_n}_{(\boldsymbol{Z}_n^*,\cdot) \textit{ is} \text{ a group!}} = \{\, a \in \boldsymbol{Z}_n \mid \gcd(a, n) = 1 \,\}$$

Not invertible	Invertible
$2, 4, 5, 6, 8$	$1, 3, 7, 9$

$\boldsymbol{Z}_{10}^+ = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$2 \cdot 1 = 2 \mod 10$
$2 \cdot 2 = 4 \mod 10$
$2 \cdot 3 = 6 \mod 10$
$2 \cdot 4 = 8 \mod 10$
$2 \cdot 5 = 0 \mod 10$
$2 \cdot 6 = 2 \mod 10$
$2 \cdot 7 = 4 \mod 10$
$2 \cdot 8 = 6 \mod 10$
$2 \cdot 9 = 8 \mod 10$

$1 \cdot 1 = 1 \mod 10$

$3 \cdot 7 = 21 = 1 \mod 10$

$9 \cdot 9 = 81 = 1 \mod 10$

$10 = 2 \cdot 5$

$2 = 2$
$4 = 2 \cdot 2$
$5 = 5$
$6 = 2 \cdot 3$
$8 = 2 \cdot 2 \cdot 2$

$\boldsymbol{Z}_{10}^* = \{1,3,7,9\}$

Euler's $\phi(n)$ function

- $\phi(n) \overset{\text{def}}{=} |\mathbf{Z}_n^*| = |\{\, a \in \mathbf{Z}_n \mid \gcd(a,n) = 1 \,\}|$

$$\overbrace{1 \cdot p,\ \ 2 \cdot p,\ \ 3 \cdot p,\ \ \cdots\ \ (q-1)\cdot p}^{q-1}$$

$$\underbrace{1 \cdot q,\ \ 2 \cdot q,\ \ 3 \cdot q,\ \ \cdots\ \ (p-1)\cdot q}_{p-1}$$

- $\phi(p) = p - 1$

- $\phi(p \cdot q) = (p-1)\cdot(q-1)$

$\phi(pq) =$ #numbers less than pq $-$ #numbers less than pq with $\gcd(x, pq) \neq 1$

$$= (pq - 1) \ - \ (q - 1 + p - 1)$$

$$= pq - q - p + 1$$

$$= (p-1)\cdot(q-1)$$

- **Note:** $\phi(n) \approx n - 2\sqrt{n} \approx n$
  - i.e.: *almost all* integers are invertible for large p, q

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\phi(n)$	**1**	**1**	**2**	**2**	**4**	**2**	**6**	**4**	**6**	**4**	**10**	**4**	**12**	**6**	**8**

Euler's Theorem

Theorem: if $(G,\circ)$ is a finite group, then for all $g \in G$:

$$g^{|G|} = e$$

- $(\boldsymbol{Z}_p^*, \cdot)$: $|\boldsymbol{Z}_p^*| = p - 1 \quad e = 1$

Fermat's theorem: if p is prime, then for all $a \neq 0 \pmod p$:

$$a^{p-1} \equiv 1 \pmod p$$

- $(\boldsymbol{Z}_n^*, \cdot)$: $|\boldsymbol{Z}_n^*| = \phi(n) \quad e = 1$

Euler's theorem: for all positive integers n, if $\gcd(a, n) = 1$ then

$$a^{\phi(n)} \equiv 1 \pmod n$$

Textbook RSA

$$\text{RSA.Enc}: \overbrace{\boldsymbol{Z}^{+} \times \boldsymbol{Z}^{*}_{\phi(n)}}^{\mathcal{PK}} \times \overbrace{\boldsymbol{Z}^{*}_{n}}^{\mathcal{M}} \to \overbrace{\boldsymbol{Z}^{*}_{n}}^{\mathcal{C}}$$

$$\text{RSA.Dec}: \underbrace{\boldsymbol{Z}^{*}_{\phi(n)}}_{\mathcal{SK}} \times \underbrace{\boldsymbol{Z}^{*}_{n}}_{\mathcal{C}} \to \underbrace{\boldsymbol{Z}^{*}_{n}}_{\mathcal{M}}$$

Enc$(pk = (n, e), M \in \boldsymbol{Z}^{*}_{n})$

1. $C \leftarrow M^{e} \mod n$
2. **return** C

pk

C

KeyGen

1. $p, q \xleftarrow{\$}$ two random prime numbers
2. $n \leftarrow p \cdot q$
3. $\phi(n) = (p-1)(q-1)$
4. **choose** e such that $\gcd(e, \phi(n)) = 1$
5. $d \leftarrow e^{-1} \mod \phi(n)$
6. $sk \leftarrow d \quad pk \leftarrow (n, e)$
7. **return** (sk, pk)

Dec$(sk = d, C \in \boldsymbol{Z}^{*}_{n})$

1. $M \leftarrow C^{d} \mod n$
2. **return** M

Common choices of e: 3, 17, 65 537

11_2 10001_2 $1\,0000\,0000\,0000\,0001_2$

Textbook RSA – correctness

Theorem: if $(G,\circ)$ is a finite group, then for all $g \in G$:

$$g^{|G|} = e$$

Euler's theorem: for all $a \in \boldsymbol{Z}_n^*$

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

Corollary I: $a^i = a^{i \bmod |G|} = a^{i \bmod \phi(n)}$

$$\mathrm{Dec}\big(sk, \mathrm{Enc}(pk, M)\big) = M$$

$$d = e^{-1} \bmod \phi(n) \iff ed = 1 \bmod \phi(n)$$

$$C^d = M^{ed} = M^{ed \bmod \phi(n)} = M^1 = M \bmod n$$

Fact: RSA also works for $M \in \boldsymbol{Z}_n$

KeyGen	
1.	$p, q \overset{\$}{\leftarrow}$ two random prime numbers
2.	$n \leftarrow p \cdot q$
3.	$\phi(n) = (p-1)(q-1)$
4.	**choose** e such that $\gcd\big(e, \phi(n)\big) = 1$
5.	$d \leftarrow e^{-1} \bmod \phi(n)$
6.	$sk \leftarrow d \qquad pk \leftarrow (n, e)$
7.	**return** (sk, pk)

Enc$(pk = (n,e), M \in \boldsymbol{Z}_n^*)$	
1.	$C \leftarrow M^e \bmod n$
2.	**return** C

Dec$(sk = d, C \in \boldsymbol{Z}_n^*)$	
1.	$M \leftarrow C^d \bmod n$
2.	**return** M

Textbook RSA – security

- Textbook RSA is *not* IND-CPA secure!
    - Deterministic
    - Malleable

- Many other attacks as well*

- Textbook RSA is *not* an encryption scheme!

- So what is it? Answer: a *one-way (trapdoor) permutation*

- * https://crypto.stackexchange.com/questions/20085/which-attacks-are-possible-against-raw-textbook-rsa

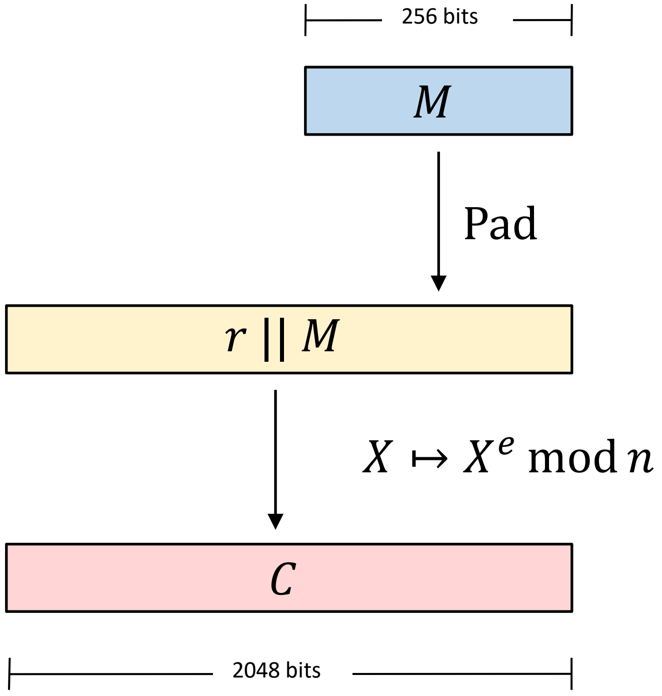
RSA in practice

- Textbook RSA is deterministic $\Longrightarrow$ cannot be IND-CPA secure

- How to achieve IND-CPA, IND-CCA?
  - *pad* message with random data before applying RSA function
  - PKCS#1v1.5 (RFC 2313)
  - RSA-OAEP (RFC 8017)

- RSA encryption is not used much in practice anymore

- RSA digital signatures still very common



Hard problems

- RSA problem (RSA): given $pk = (e, n)$ and $C = M^e \bmod n$
  find M

- Factoring problem (FACT): given $n = pq$ find p and q

- FACT $\geq$ RSA

Demo RSA encryption

- Demonstration using SageMath

- https://sagecell.sagemath.org/

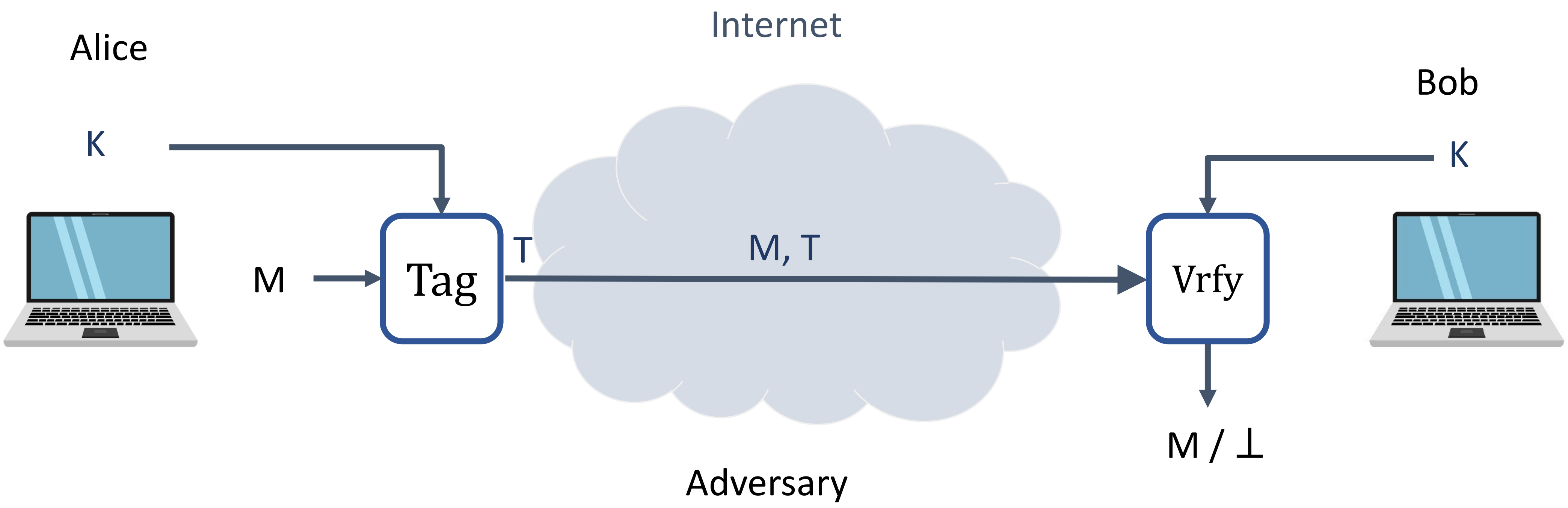
A short summary

- We can build IND-CPA secure ElGamal scheme based on DDH assumption

- Padding with randomness, we can transfer Textbook RSA to IND-CPA scheme

Digital Signature

Achieving integrity: MACs

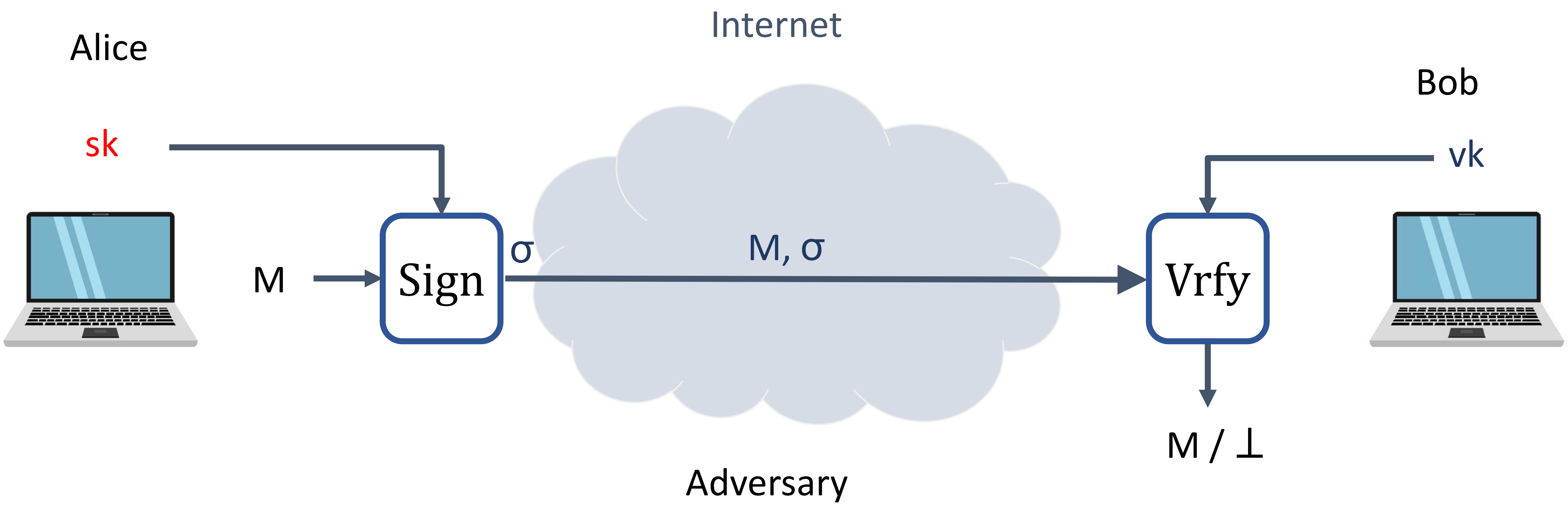


Tag : tagging algorithm (public)

K : tagging / verification key (secret)

Vrfy: verification algorithm (public)

Achieving integrity: digital signatures



Sign : tagging algorithm (public)

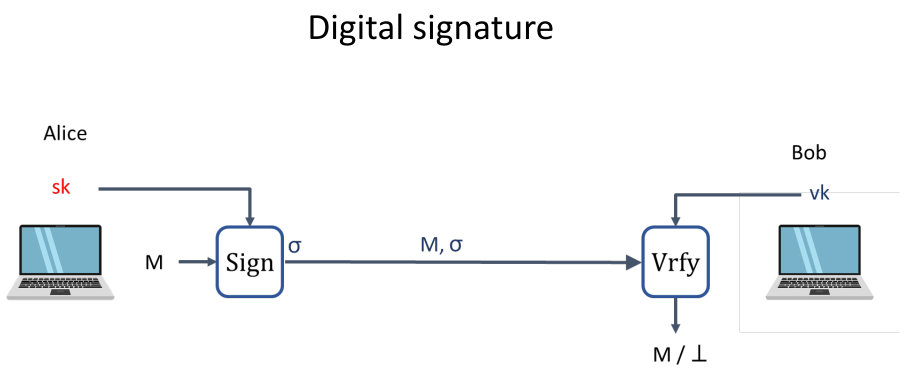
sk : signing key (secret)

Vrfy : verification algorithm (public)

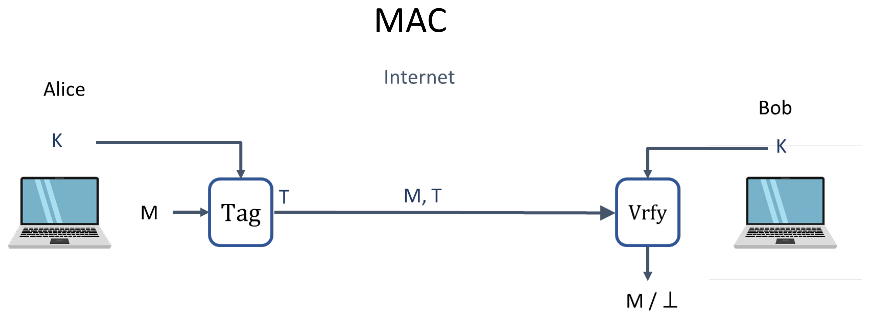
vk : verification key (public)

Digital signatures vs. MACs

- Digital signatures can be verified by *anyone*



- MACs can only be verified by party sharing the same key



- **Non-repudiation:** Alice cannot deny having created σ
  - But she can deny having created T (since Bob could have done it)

Digital signatures – syntax

A **digital signature** scheme is a tuple of algorithms $\Sigma = (\text{KeyGen}, \text{Sign}, \text{Vrfy})$

$\text{KeyGen} : (\,) \to \mathcal{SK} \times \mathcal{VK}$

$\text{Sign} : \mathcal{SK} \times \mathcal{M} \to \mathcal{S}$

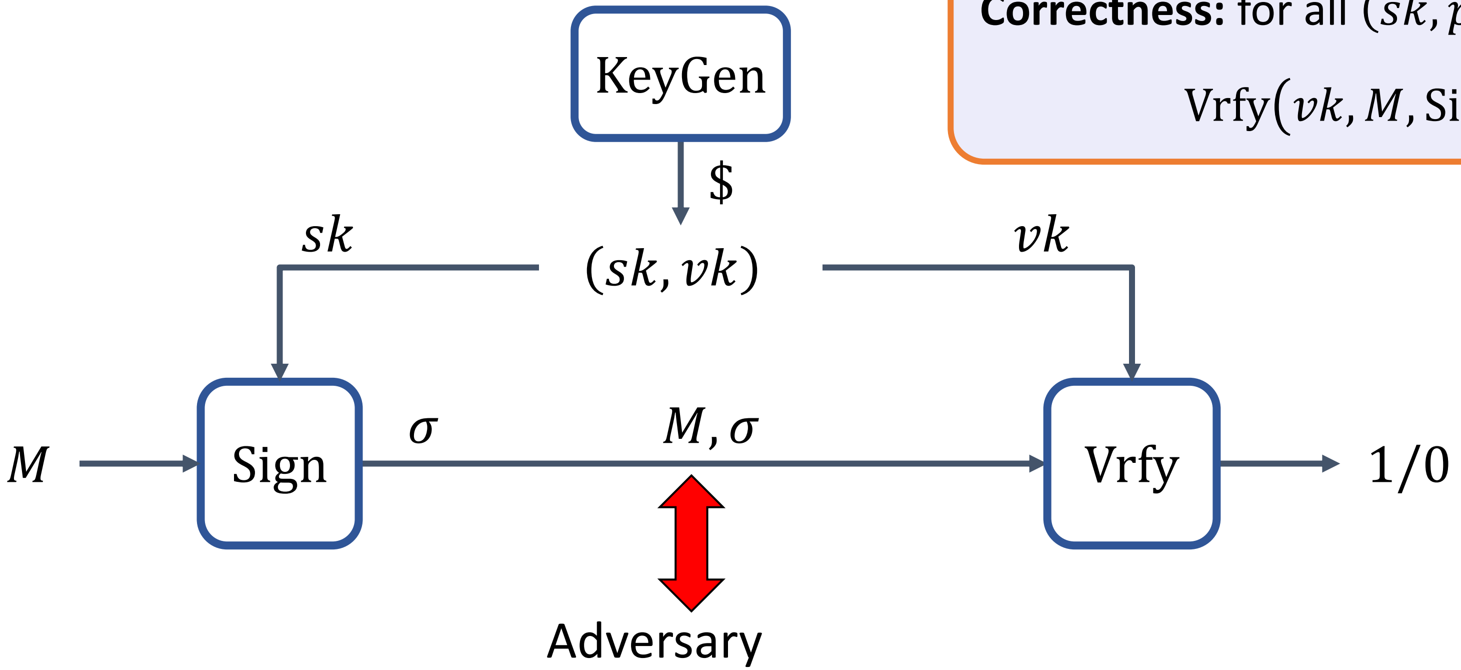
$\text{Sign}(sk, M) = \text{Sign}_{sk}(M) = \sigma$

$\text{Vrfy} : \mathcal{VK} \times \mathcal{M} \times \mathcal{S} \to \{0,1\}$

$\text{Vrfy}(vk, M, \sigma) = \text{Vrfy}_{vk}(M, \sigma) = 1/0$

Correctness: for all $(sk, pk) \leftarrow \text{KeyGen}$:

$$\text{Vrfy}\big(vk, M, \text{Sign}(sk, M)\big) = 1$$



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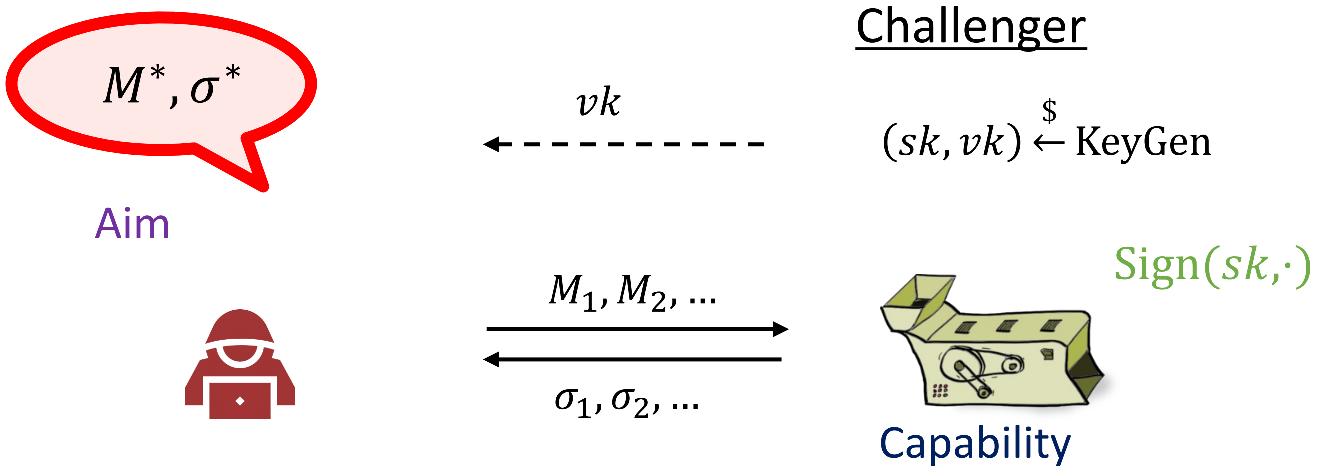
Digital signatures – security: UF-CMA

$\mathbf{Exp}_{\Sigma}^{\text{uf-cma}}(A)$

$(sk, vk) \xleftarrow{\$} \Sigma.\mathrm{KeyGen}$
$S \leftarrow [\,]$
$(M^*, \sigma^*) \leftarrow A^{\mathrm{SIGN}_{sk}(\cdot)}(vk)$
if $\Sigma.\mathrm{Vrfy}(vk, M^*, \sigma^*) = 1$ and $M \notin S$ **then**
    return 1
 else
    return 0

$\mathrm{SIGN}_{sk}(M)$

$\sigma \leftarrow \Sigma.\mathrm{Sign}(sk, M)$
$S.\mathrm{add}(M)$
return σ



If σ^* is a valid signature for M^* (not asked before) then the adversary has **forged** a signature

Definition: The **UF-CMA-advantage** of an adversary A is

$$\mathbf{Adv}_{\Sigma}^{\text{uf-cma}}(A) = \Pr\left[\mathbf{Exp}_{\Sigma}^{\text{uf-cma}}(A) \Rightarrow 1\right]$$

Textbook RSA signatures

$$\text{RSA.Sign: } \overbrace{\boldsymbol{Z}^{+} \times \boldsymbol{Z}^{*}_{\phi(n)}}^{\mathcal{SK}} \times \overset{\mathcal{M}}{\boldsymbol{Z}^{*}_{n}} \rightarrow \overset{\mathcal{S}}{\boldsymbol{Z}^{*}_{n}}$$

$$\text{RSA.Vrfy: } \underbrace{\boldsymbol{Z}^{+} \times \boldsymbol{Z}^{*}_{\phi(n)}}_{\mathcal{PK}} \times \underset{\mathcal{M}}{\boldsymbol{Z}^{*}_{n}} \times \underset{\mathcal{S}}{\boldsymbol{Z}^{*}_{n}} \rightarrow \{1,0\}$$

KeyGen

```
p, q ←$ two random prime numbers
n ← p · q
φ(n) = (p − 1)(q − 1)
choose e such that gcd(e, φ(n)) = 1
d ← e^{-1} mod φ(n)
sk ← (n, d)        vk ← (n, e)
return (sk, vk)
```

Vrfy$(vk = (n, e), M \in \boldsymbol{Z}^{*}_{n}, \sigma)$

```
if σ^e = M mod n then
    return 1
else
    return 0
```

vk

M, σ

Sign$(sk = (n, d), M \in \boldsymbol{Z}^{*}_{n})$

```
σ ← M^d mod n
return σ
```

$$d = e^{-1} \bmod \phi(n) \iff ed = 1 \bmod \phi(n)$$

$$\sigma^{e} = M^{de} = M^{ed \bmod \phi(n)} = M^{1} = M \bmod n$$

Insecurity of Textbook RSA signature

Given $\sigma_1 = M_1^d, \sigma_2 = M_2^d$

$\sigma_1\sigma_2 = (M_1M_2)^d mod\ n$ is a signature of $M_1M_2\ mod\ n$

Many other attacks exist

Hash-then sign paradigm

$$\text{RSA.Sign}: \overbrace{\boldsymbol{Z}^{+} \times \boldsymbol{Z}^{*}_{\phi(n)}}^{\mathcal{SK}} \times \overbrace{\{0,1\}^{*}}^{\mathcal{M}} \rightarrow \overbrace{\boldsymbol{Z}^{*}_{n}}^{\mathcal{S}}$$

$$\text{RSA.Vrfy}: \underbrace{\boldsymbol{Z}^{+} \times \boldsymbol{Z}^{*}_{\phi(n)}}_{\mathcal{PK}} \times \underbrace{\{0,1\}^{*}}_{\mathcal{M}} \times \underbrace{\boldsymbol{Z}^{*}_{n}}_{\mathcal{S}} \rightarrow \{1,0\}$$

KeyGen

```
p, q ←$ two random prime numbers
n ← p · q
φ(n) = (p − 1)(q − 1)
choose e such that gcd(e, φ(n)) = 1
d ← e^{−1} mod φ(n)
sk ← (n, d)      vk ← (n, e)
return (sk, vk)
```

Vrfy$(vk = (n, e), M \in \boldsymbol{Z}^{*}_{n}, \sigma)$

```
if σ^e = H(M) mod n then
    return 1
else
    return 0
```

$\xleftarrow{\quad vk \quad}$

$\xleftarrow{\quad M, \sigma \quad}$

$$H : \{0,1\}^{*} \rightarrow \boldsymbol{Z}^{*}_{n}$$

Sign$(sk = (n, d), M \in \boldsymbol{Z}^{*}_{n})$

```
σ ← H(M)^d mod n
return σ
```

Discrete-log-based signatures: (EC)DSA

- Schnorr
  - Elegant design
  - Has formal security proof (based on DLOG problem and H assumed perfect)
  - Patented (expired in February 2008)

- (EC)DSA
  - Non-patented alternative
  - Derived from ElGamal-based signature scheme
  - More complicated design than Schnorr
  - No security proof
  - Standardized by NIST
  - Very widely used

A short summary

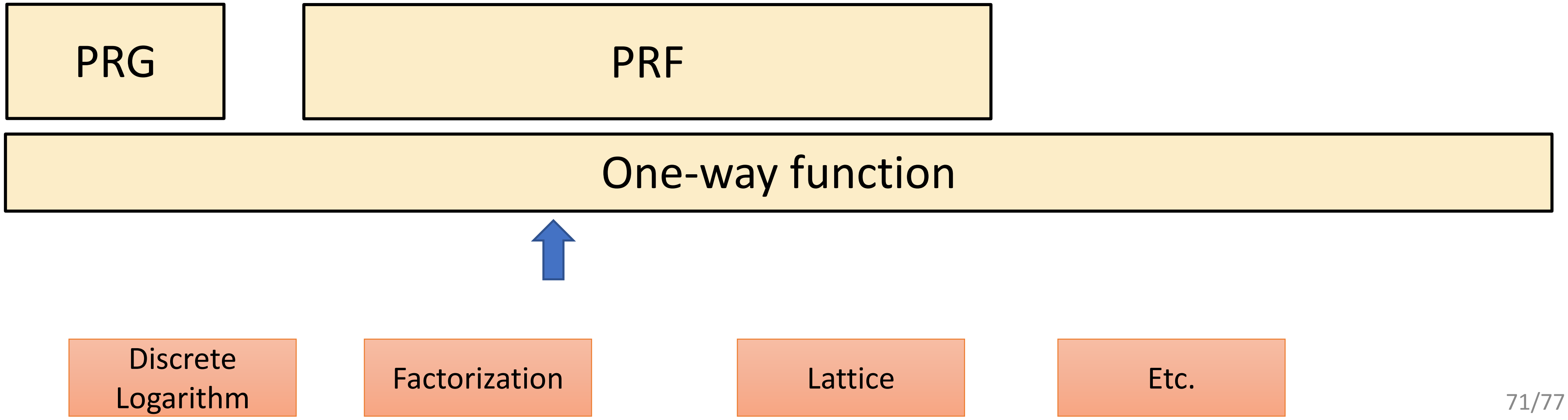
- Hash-then sign paradigm of RSA gives a secure signature

- There are Discrete-log-based signatures, ECDSA, and Schnorr

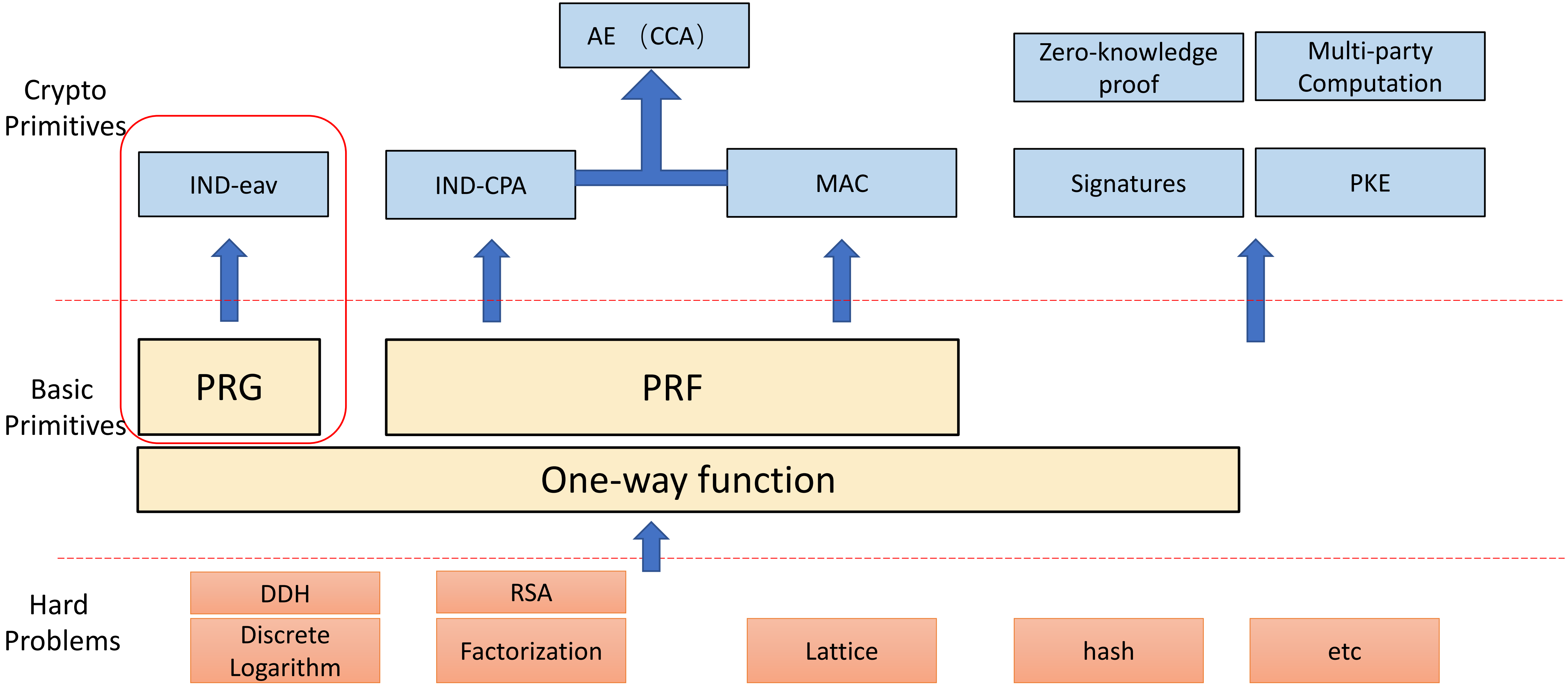
One more thing

- We leave the construction of Pseudorandom generator (PRG) and Pseudorandom function (PRF) in lecture 2

- One-way function f: given $y = f(x)$ for random x, it is hard to find x' such that $y = f(x')$



Big picture of Cryptography



Primitive	Functionality + syntax	Hardness assumption	Security	Examples
Diffie-Hellman	Derive shared value (key) in a cyclic group $A^b = g^{ab} = B^a$	Discrete logarithm (DLOG) Decisional Diffie-Hellman (DDH)		$(\boldsymbol{Z}_p^*, \cdot)$ −DH $(E(\boldsymbol{F}_p), +)$ −DH
RSA function	One-way trapdoor function/permutation	Factoring problem RSA-problem		Textbook RSA
Public-key encryption	Encrypt variable-length input $\mathrm{Enc} : \mathcal{PK} \times \mathcal{M} \rightarrow \mathcal{C}$	Decisional Diffie-Hellman (DDH) Factoring problem RSA-problem	IND-CPA IND-CCA	ElGamal Padded RSA
Digital signatures	$\mathrm{Sign} : \mathcal{SK} \times \mathcal{M} \rightarrow \mathcal{S}$ $\mathrm{Vrfy} : \mathcal{VK} \times \mathcal{M} \times \mathcal{S} \rightarrow \{1,0\}$	RSA-problem Discrete logarithm (DLOG)	UF-CMA	Hashed-RSA ECDSA Schnorr

Thank you